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An investigation of the sample sizes required for estimating line lengths of single server queues using simulation

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AN INVESTIGATION OF THE SAMPLE SIZES REQUIRED FOR ESTIMATING LINE LENGTHS OF SINGLE SERVER QUEUES USING SIMULATION

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ABSTRACT

An investigation of the sample sizes required in order to find average queue lengths with specified accuracy is presented in this thesis. The queuing systems studied are single server models with infinite queues and no priorities. Estimates of the sample sizes are developed mathematically for non-exponential interarrivals, non-exponential service times, and for the case where both service times and interarrival times are non-exponential. The estimates were compared with results obtained by simulating the queuing systems.

The simulation was written in FORTRAN IV language for the IBM-360 Model 50. Weibull and normally distributed interarrival and service times were used for the runs. Seven values of coefficient of variation for the Weibull distribution were tested, of which three were less than one and four greater than one. The three coefficients of variation used for the normal distribution were less than one.

The results indicate that non-exponential interarrival and non-exponential service time systems can be estimated by the same method. The properties of the system were both non-exponential service and interarrival times are present are estimates less accurately, but the estimates may still be useful for planning of simulation experiments.

I INTRODUCTION

Monte Carlo simulation has become a widely used tool of Operations Research. It is applied in the study of a variety of models where other methods of analysis cannot be applied or would be extremely difficult to apply. Many models encountered by analysts today fall into these categories and simulation is used to study their operation.

Situations where simulation has been used profitably include the following:

(1) Evaluation of adequacy of an existing model.

(2) Evaluation of the design of a proposed system.

(3) To find the best of several alternative ways of operating an existing system.

(4) Solution of mathematical equations which do not yield to other kinds of analysis.

The use of Monte Carlo Simulation involves two major steps to obtain useful results. The first is construction of a model which adequately describes the process under study. The second problem, usually referred to as tactical, is obtaining statistical data from the model. Model construction has been the focus of attention in many papers on simulation. Models of business firms, queuing networks; inventory systems, scheduling rules, and many other problems have been discussed in the literature[18]. The process of constructing a model which is capable of describing the system operation is often a major undertaking. Connected with this activity is the problem of obtaining values of parameters which characterize the model. Data collection and statistical analysis may be required here. If the study is being conducted on a proposed system which does not presently exist, no data will be available. In this case, the parameters must be predicted or assumed.

Tactical problems are concerned with simulated operation of the model in order to obtain useful information about its behavior. First, input data must be obtained in some fashion. As stated above, real data is often not available and input data must be generated. Various means of generating random numbers have been derived and these are widely employed. Once a method for generation of input is found, the next problem is constructing an experiment that will provide suitable information about the model. The methods of experimental design and sampling may be applied here. Since the input is controlled by the experimenter there are several methods of constructing simulation experiments which will reduce the total sampling required. Some of these methods are stratified sampling, use of the same random numbers to compare alternatives, and sampling with replacement[11]. The objective of these schemes is to be able to test hypothesis about the characteristics of the model with as few samples as possible.

One of the practical problems encountered in simulation studies is the cost of computer time (or calculation effort) to achieve sufficiently accurate results. It would be desirable if this could be estimated in advance of simulation runs. The difficulty in doing this is that the run length must be estimated in advance. It will be the purpose of this thesis to investigate the problem of estimating the lengths of samples needed in the study of single server queuing systems. Single server systems are selected because of their simplicity and because the results may have use in analyzing more complex queuing systems. The variable observed will be the total number of units in the line and in service. A common criteria for accuracy of the results in a simulation of this nature is to require that

$$\Pr\left(\left|\frac{\hat{L} - L}{L}\right| \leq \alpha\right) \geq \beta$$

where

$\hat{L}$ = the observed value of the average line length.

L = the true value of average line length.

α = Accuracy ratio desired.

β = confidence desired.

This criteria will be used in the determination of the required sample lengths for this thesis.

Often the experimenter will not have exact knowledge of the type of arrival and service distributions that will be used in the simulation, necessitating the use of empirical distributions. In all cases, the experimenter will know the mean and variance of the distributions. Because the distributions are sometimes unknown in mathematical form, the formulas derived for estimating the sample sizes in this thesis will use only the mean and variance as parameters. This will introduce some error, but will keep the calculations easy enough to be useful.

II BACKGROUND

The use of simulation as an analytic tool has its start in the founding of probability theory. Sampling experiments were constructed by Karl Pearson to verify or discover the distribution of certain statistical processes[22]. In these experiments, random numbers were generated by drawing heads or disks from an urn. Later Tippet introduced the use of random number tables and published a table of 10,400 random digits. Several random number tables were subsequently published and techniques of testing numbers for randomness were devised. These tests include the uniform probability test, poker test, and serial correlation test. A discussion of these methods are given in Tocher[22].

During the second world war, Von Neumann and Ulam developed methods of solving mathematical equations by use of random sampling. They named their method "Monte Carlo" and used it to evaluate diffusion operations. In order to generate the large quantity of random digits required, Von Neumann devised a way of generating them by use of mathematical formulas. He called this the mid-square technique. A p-digit number X_0 is squared giving a number of length 2p. The middle digits are then taken as X_1, the next number in the series. For example

$$X_0 = 37$$

$$X_0^2 = 1369$$

$$X_1 = 36$$

The objection to the method is that it is cyclic, that is the series will repeat itself after an interval of unknown length. This property led to the development of other formulas for generating random numbers. Lehemer[22] developed a method for generating random numbers which gives a predictable minimum cycle length. The numbers are generated by the integer sequence

$$X_{n+1} = k\, X_n \bmod m$$

where m and k are arbitrary integers. The best choice of M will depend upon the computing technique or machine used. For binary machines m is usually a power of two, less one, that is $m = 2^{p-1}$. This gives a cycle length of 2^{p-2} if k is prime and X_0 odd.

With the coming of the digital computer, simulation has become an increasingly popular analytical tool. The immense amount of computational effort required in the course of a simulation could now be handled by the computer in a reasonable amount of time and at reasonable costs. At the same time, many of the stochastic models developed in recent years are of a nature that mathematical analysis of them approaches impossibility. Thus simulation has been required to obtain quantitative information about their behavior. As computers become more economical and models more complicated, the use of simulation is expected to become more frequently used.

There have been two investigations of sample length of simulation experiments which have appeared in the literature. One was done by M. E. Brenner[3] to find the length of sample required for study of single inventory systems. The subject of interest was the cost penalty associated with sampling error. Two inventory models were selected for study, one a periodic review model and the other a reorder point model. Poisson demand distributions were used for both models. The optimal solution to both of these models is given in the literature of inventory theory, hence the solution by simulation could be compared to the known true solution. The vector of management controlled variables such as order quantity, safety stock, etc., are designated U with resulting cost C(U). The optimal true policy is U_0 and the optimal policy found by simulation is U_0'. U_0' differed from U_0 due to sampling error. Since U_0' is not generally the true optimum there is a cost penalty W associated with U_0'. That is

$$W = C(U_0') - C(U_0)$$

$C(U_0')$ is always greater than $C(U_0)$ and therefore $W \geq 0$. The values of W were observed for various values of sample size n and for various operating characteristics. The following functional relationship was found to fit the data.

$$W = \frac{A}{n^k}$$

where

A = Value of W at n=1.

n = Number of periods.

k = A parameter found by simulation.

The variables A and k are functions of the model structure and operating parameters. Graphical relationships were determined for A in terms of inventory holding costs, ordering costs, average lead time, and unit cost for each model. The variable k was found to be essentially constant for a model type. The relationship permits the calculation of the sample size needed to obtain a given average cost penalty W. This allows an estimate of simulation costs to be made in advance of the computer runs. The problem is that the models are solvable by mathematical techniques, and it is not known how to extend the results to models which are difficult to solve mathematically.

Geisler[9,10] also studied the sample size relationships for inventory models using a somewhat different approach. First the statistical properties of the inventory models were determined mathematically. Two types were solved, both reorder point models. One of these had a zero procurement lead time and the other had a positive lead time. The items of interest were the quantity of inventory on hand, the shortages, and the overages at the end of each period n. Formulas for the variances and serial correlations of these variables were computed. Using these expressions, the sample sizes necessary to find each of the three variables with a stated confidence and precision were computed. The predicted results were then compared to the actual results obtained by simulation. Agreement was quite close and the feasibility of using mathematical expressions to predict sample size was established.

There are two methods of statistical analysis of simulation sequences that are mentioned in the literature[6,12]. These are classified as the serial correlation and sub-group methods. In serial correlation the variance of a sample reference X_t is estimated by treating the simulation as one long experiment. The formulas for estimating the mean and variance are

$$\bar{X} = \sum_{t=1}^{T} X_t/T$$

$$\hat{V} = \text{var}\ (\bar{X}) = \frac{1}{T}\left[R_0 + 2\sum_{\tau=1}^{M}(1 - \frac{\tau}{M})R_\tau\right],\quad M < T-1$$

$$\hat{R}_\tau = \frac{1}{T-\tau}\sum_{t=1}^{t-\tau}(X_t - \bar{X})(X_{t+\tau} - \bar{X})$$

The value of M used in these formulas must be large enough that the variance estimate $\hat{V}$ does not change appreciably if a larger M is used. Also for a Gaussian sequence the distribution of $\hat{V}/V$ may be approximated by a multiple of chi-square with equivalent degrees of freedom

$$EDF(0) = \frac{1.5\ T}{M}$$

Hence it is desirable to have the ratio M/T approach zero as T increases so that sufficient degrees of freedom for statistical testing are obtained. On the other hand, M must be large enough that V is not underestimated by a gross amount.

In the sub-group method of analysis, the collection of samples is divided into groups and a mean for each group is computed. These means are then treated as individual independent samples. The number of sample points in each sub-group must of course be large enough that the sub-group means are virtually independent. The following relations are used for analysis.

For the N sub-group means of sample size M

$$\bar{X}_{N+1} = \frac{1}{M} \sum_{i=NM+1}^{M(N+1)} X_i$$

The grand mean is

$$\bar{X} = \sum_{i=1}^{N} X_i$$

The variance of $\bar{X}$ is then

$$\mathrm{Var}\,(\bar{X}) = \sum_{i=1}^{N} (X_i - \bar{X})^2/N$$

The statistical tests are then performed as though there are N independent samples.

A comparison of the two methods shows that the sub-group method requires less computational effort for each sample. The running time is therefore usually less than for the serial correlation technique. The serial correlation method has the advantage that no assumptions about independence need to be made. It also provides information about the autocorrelation structure, which is usually of interest to the experimenter.

Methods of Determining Starting Bias

When performing simulation experiments it is common practice to begin operation of the model empty and idle. This has to be done because there is usually no estimate available of steady state conditions. Assuming a 'reasonable' set of conditions may help to reduce the length of the total run, but one must be careful that the initial assumed conditions have worn off before collecting data. In either case the problem of determining when the system reaches steady state has to be resolved.

So far no methods of determining in advance the point at which the process reaches steady state has been developed. A method often used in the course of a simulation is to plot the data and observe the trend in the state of the system. The point where sampling begins is assumed to be where the system appears to have reached steady state. The exact point chosen is somewhat arbitrary and will vary with the judgement and inclination of the individual making the decision.

Methods have also been developed where the restart point is determined by statistical methods. One way is to compare the state probability distribution at two points during the simulation[15]. If a statistical test cannot reject the hypothesis that the two distributions are the same, then the system is considered to be in steady state. If the hypothesis is rejected, then further simulation and testing is required. This procedure will result in a relatively lengthy run in period due to the fact that a reasonably good estimate of all state probabilities must be obtained.

The purpose of this study is to obtain estimates of the run times in advance and this includes the transient time. The next chapter will discuss the problem of estimating the transient time in advance for a simple queuing system.

III DEVELOPMENT OF THE MODEL

As mentioned in the last chapter there are two common methods of obtaining estimates of the accuracy of line length L. These are the serial correlation and sub-group methods. To find estimates of the sample size for either of these methods requires knowledge of the autocorrelation function of the time series. Since the autocorrelation function must be observed in either case, the serial correlation method will be used in this study. The estimate of L is the same in both cases, but the estimate of accuracy for the serial correlation represents more information about the process than does the sub-group method. To estimate the sample length, the following must be obtained:

(1) Variance of the line length.

(2) Autocorrelation of the samples.

(3) The transient behavior of the queuing system.

(4) The required accuracy of the result.

The autocorrelation function of a simple queue will be obtained in the next section.

Autocorrelation Function of the Queue Length for a M/M/1 System

The autocorrelation function of a time series is defined as

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t)\, X(t+\tau)\, dt \qquad (3.1)$$

For a M/M/1 queuing system where the time series being observed is the time length L(t) this expression becomes

$$R(\tau) = \lim_{T \to \infty} \frac{I}{T} \int_0^T L(t)L(t+\tau)dt \qquad (3.2)$$

Let p_i be the probability that the system is in state i, and let $p_{in}(t)$ be the probability that the system is in state n at t units of time after starting at state i. The autocorrelation expression then becomes

$$R(\tau) = \sum_{i=0}^{\infty} i\, p_i \sum_{n=0}^{\infty} n\, p_{in}(\tau) \qquad (3.4)$$

The following expressions for p_i and $p_{in}(\tau)$ are given in Saaty[21]:

$$p_i = \rho^i(1-\rho)$$

$$p_{in}(\tau) = e^{-(\lambda+\mu)\tau}\Big[(\sqrt{\lambda\mu})^{i-n}\, I_{n-i}(2\sqrt{\lambda\mu}\,\tau) + (\sqrt{\tfrac{\mu}{\lambda}})^{i-n+1} \qquad (3.5)$$

$$I_{n+i+1}(2\sqrt{\lambda\mu}\,\tau) + (1-\tfrac{\lambda}{\mu})(\tfrac{\lambda}{\mu})^n \sum_{k=n+i+2} (\tfrac{\mu}{\lambda})^k I_k(2\sqrt{\lambda\mu}\,\tau)\Big]$$

where I is the modified Bessel function of the first kind.

The resulting expression for $R(\tau)$ becomes

$$R(\tau) = \frac{\lambda^2}{(\lambda-\mu)^2} + (\lambda-\mu)\frac{\lambda\mu}{\pi}\int_0^{2\pi} \sin^2\theta\, \frac{e^{-w\tau}}{w^3}\, d\theta \qquad (3.6)$$

where

$$w = \mu + \lambda - 2\sqrt{\lambda\mu}\cos\theta$$

Saaty[21] also gives an approximation of the above formula for $R(\tau)$. It is

$$R(\tau) \simeq \frac{\lambda^2}{(\mu-\lambda)^2} + \frac{\lambda\mu}{(\mu-\lambda)^2} \exp\left[\frac{-(\mu-\lambda)^2}{\lambda}\tau\right] \tag{3.7}$$

The autocovariance function $A(\tau)$ is given as

$$A(\tau) = R(\tau) - L^2 \tag{3.8}$$

or

$$A(\tau) = \frac{\lambda^2}{(\mu-\lambda)} + \frac{\lambda\mu}{(\mu-\lambda)^2} \exp\left[\frac{-(\mu-\lambda)^2_{\tau}}{\lambda}\right] - \frac{\lambda^2}{(\mu-\lambda)^2} \tag{3.9}$$

$$A(\tau) = \frac{\mu\lambda}{(\mu-\lambda)^2} \exp\left[\frac{-(\mu-\lambda)^2}{\lambda}\tau\right] \tag{3.10}$$

at $\tau = 0$ this expression becomes

$$A(0) = \frac{\mu\lambda}{(\mu-\lambda)^2} \tag{3.11}$$

This agrees with the expression for the variance of the line length as it should.

The correlation coefficient of $r(\tau)$ is then

$$r(\tau) = \frac{A(\tau)}{A(0)} \tag{3.12}$$

$$r(\tau) = \exp\left[\frac{-(\mu-\lambda)^2}{\lambda}\tau\right] \tag{3.13}$$

or in terms of $\rho = \lambda/\mu$

$$r(\tau) = \exp\left[\frac{-(1-\rho)^2}{\rho^2}\lambda\tau\right] \tag{3.14}$$

If the system is sampled at multiples of the average service time $1/u$, the expression becomes

$$r(k) = \exp \frac{-(1-\rho)^2}{\rho} k \tag{3.15}$$

where k is the number of multiples of the service time between samples.

<u>Bias</u>

In the course of simulation experiments it is often necessary to begin an experiment in an empty state. This is usually due to lack of knowledge of the steady state conditions. Consequently the results will be biased by this assumption. Customarily this problem is overcome by letting the system run until it appears that a steady state condition is reached. To predict the length of time the system must run before reaching steady state, the transient behavior of the queue must be examined. P. Morse[17] has solved for the transient behavior of single server exponential queues with finite waiting lines. The time dependent equations for this system are

$$\begin{aligned} \dot{p}_n &= \mu p_{n+1} + \lambda p_{n-1} - (\lambda+\mu) p_n \\ \dot{p}_0 &= \mu p_1 - \lambda p_0 \\ \dot{p}_N &= \mu p_{N-1} - \lambda p_N \end{aligned} \tag{3.16}$$

This set of equations can be solved by methods of differential equations. Let E_{mn} be the rate of transition from state m to state n. The operation for the rate of change of p_m is then

$$\frac{dp_m}{dt} = \sum_{n=0}^{N} E_{mn}\, p_n \tag{3.17}$$

Solutions are obtained by setting

$$p_m(t) = \sum_s B_{ms}\, e^{-\gamma_s t} \tag{3.18}$$

where γ_s are the solutions to the equation

$$\begin{vmatrix} (E_{00} + \gamma_s) & E_{01} & \cdots & E_{0N} \\ E_{10} & (E_{11} + \gamma_s) & \cdots & E_{1N} \\ \vdots & \vdots & & \vdots \\ E_{N0} & E_{N1} & \cdots & (E_{NN} + \gamma_s) \end{vmatrix} = 0 \tag{3.19}$$

Letting $p_n = \rho^{\frac{1}{2}n} B_{n,s} e^{-\gamma_s t}$ and $\gamma_s = \mu X_s$ we obtain

$$\sqrt{\rho}\, B_{1,s} + (X_s - \rho)\, B_{0,s} = 0 \tag{3.20}$$

$$\sqrt{\rho}\, (B_{n+1,s} + B_{n-1,s}) + (X_s - 1 - \rho) = 0$$

$$\sqrt{\rho}\, B_{N-1,s} + (X_s - 1)\, B_{N,s} = 0$$

Letting $B_{n,s} = \sin(ny) - \sqrt{\rho}\ \sin(n+1)y$ and making $\sin(N+1)y=0$ the above equations reduce to a single equation

$$X_s = \rho + 1 - 2\sqrt{\rho}\ \cos y \tag{3.21}$$

$\sin(N+1)y$ can be made zero by letting $y = s\pi/(N+1)$ where s is one of the integers 1,2,...,N. In fact s is the second subscript of $B_{n,s}$ and the subscript of $X_s = (\gamma_s/\mu)$.

The N solutions of the equations are

$$p_n(t) = p_n + \rho^{\frac{1}{2}n} \sum_{s=1}^{N} C_s \left[\sin \frac{sn\pi}{N+1} - \sqrt{\rho} \sin \frac{s(n+1)\pi}{N+1} \right] e^{-\gamma_s t}$$

(3.22)

$$\gamma_s = \lambda + \mu - 2\sqrt{\lambda\mu} \cos\left(\frac{s\pi}{N+1}\right) = \mu X_s \quad \begin{matrix}(s=1,2,\ldots,N)\\(n=0,1,\ldots,N)\end{matrix}$$

where the coefficients are chosen to fit the initial values of the p's at t = 0. These can be found from the boundary conditions $p_n(0) = \delta_{nm}$ where m is the number initially in the system and δ the delta function. The time dependent solutions found by this procedure are

$$p_n^m(t) = p_n + \frac{2\rho^{\frac{1}{2}(n-m)}}{N+1} \sum_{s=1}^{N} \left[\sin \frac{sm\pi}{N+1} - \sqrt{\rho} \sin \frac{(m+1)\pi}{N+1} \right]$$

$$\left[\sin \frac{sn\pi}{N+1} - \sqrt{\rho} \sin \frac{(n+1)\pi}{N+1} \right] e^{-\gamma_s t} \qquad (3.23)$$

The smallest value of γ_s will determine how long the queue will take to settle into steady state conditions. The smallest value of γ_s is γ_1 and when N is large γ_1 has a value:

$$\gamma_1 = (\sqrt{\mu} - \sqrt{\lambda})^2 \qquad (3.24)$$

In order for the transient to become small compared to the steady state component the queue must operate until $e^{-\gamma_1 t}$ becomes small. For convenience the expression can be put in terms of average service time multiples μt. This gives

$$e^{-\frac{1}{\mu}(\mu t)}$$

or

$$e^{-\frac{(\sqrt{\mu}-\sqrt{\lambda})}{\mu}(\mu t)}$$

or

$$e^{-(1-\rho-2\sqrt{\rho})\,\mu t} \tag{3.25}$$

The expression

$$Z = \frac{1}{1+\rho-2\sqrt{\rho}} \tag{3.26}$$

has units of service time multiples and is a measure of the time required for the transients to die out.

The expressions for the variance, autocorrelations, and mean line lengths will require modification for application to systems which do not have service times or arrival times which are exponential. For systems where the service times are not exponential the following expressions apply when $\rho < 1$[13]

$$P_o = 1 - \rho \tag{3.27}$$

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

$$L = \rho + L_q$$

$$W_q = L_q/\lambda$$

$$W = W_q + 1/\mu$$

where σ^2 is the variance of the service time. The expression for

L_q may be rearranged as follows

$$L_q = \frac{\frac{\lambda^2}{\mu^2}(\mu^2\sigma^2) + \rho^2}{2(1-\rho)} \tag{3.28}$$

$$L_q = \frac{\rho^2(\mu^2\sigma^2) + \rho^2}{2(1-\rho)}$$

$$L_q = \frac{\rho^2(\mu^2\sigma^2+1)}{2(1-\rho)}$$

L_q is related to the square of the variance to mean ratio $\mu^2\sigma^2$. The expressions for the variance of the line length and correlation of X_n and X_{n+1} are not available in a general formula as the one above. They must be computed for each individual case.

It is possible to obtain an approximating expression for these quantities by comparison to the exponential service time case. The expression for variance of the line length for the M/M/1 system is

$$\text{Var (L)} = \frac{\rho}{(1-\rho)^2} \tag{3.29}$$

The results of investigations by Morse[17] indicate that the variance of the line length changes in about the same proportion as the variance of the service time. The estimated variance of the line is then:

$$\text{Var (L)} = \frac{\rho}{(1-\rho)^2}\,\mu^2\sigma^2 \tag{3.30}$$

An expression for the correlation of X_n and X_{n+1} may be obtained similarly by extension of the exponential case. The formula for the correlation in the exponential system is

$$r(k) = \exp\left(-\frac{(1-\rho)^2}{\rho} k\right) \tag{3.31}$$

The coefficient of k in (3.15) is $-1/\mathrm{var}(L)$. The approximate correlation will be taken then as

$$r(k) = \exp\left[-\frac{1}{\mathrm{var}(L)} k\right] \tag{3.32}$$

where var(L) is given in (3.30).

Non-Exponential Arrivals

For non-exponential interarrivals and exponential service times, the estimates of the statistical properties is not as computationally simple as for the service time case. As will be seen, the estimates depend upon both the shape of the distribution and upon ρ.

Saaty[21] gives some results which help in the process of arriving at results for the non-exponential arrival case. These are valid for a queue with exponential service times and interarrival times which are arbitrarily and independently distributed according to some distribution a(t). The mean of this distribution will be called a. The results for these conditions are

$$\lim_{t \to \infty} p_n(t) = \begin{cases} \frac{1}{\mu a}(1-\xi_0)\,\xi_0^{n-1}, & n \geq 1 \\ 1 - \frac{1}{\mu a}, & n = 0 \end{cases} \tag{3.33}$$

is a root of the equation

$$a^*\left[s + (1-X)\mu\right] = X \tag{3.34}$$

where a*(s) is the Laplace transform of a(t). ξ is the root of

(3.34) with modulus less than one and ξ_0 is the value of $\xi(s)$ at $s = 0$. A root ξ_0 which is less than one in magnitude exists if $a < 1/\mu$. Thus the value of p_n is proportional to $\xi_0^{\,n}$ for this case. Using formulas for geometric series, the formula for line length may be obtained

$$L = \sum_{n=1}^{\infty} \frac{n}{\mu a} (1-\xi_0)\, \xi_0^{\,n-1} \tag{3.35}$$

$$L = \frac{1}{\mu a} (1-\xi_0) \sum_{n=1}^{\infty} n\, \xi_0^{\,n-1} \tag{3.36}$$

$$L = \frac{1}{\mu a(1-\xi_0)} \tag{3.37}$$

Similarly

$$\mathrm{Var}(L) = \sum_{n=1}^{\infty} n^2 p_n - L^2 \tag{3.38}$$

$$\mathrm{Var}(L) = \frac{\xi_0 + 1}{\mu a(1-\xi_0)^2} \quad \frac{1}{(\mu a)^2(1-\xi_0)^2} \tag{3.39}$$

$$\mathrm{Var}(L) = \frac{\mu a(\xi_0+1) - 1}{(\mu a)^2(1-\xi_0)^2} \tag{3.40}$$

In terms of $\rho = \frac{1}{\mu a}$

$$L = \frac{\rho}{1-\xi_0} \tag{3.41}$$

$$\mathrm{Var}(L) = \frac{\rho(\xi_0+1)-\rho^2}{(1-\xi_0)^2} \tag{3.42}$$

The state probability, mean, and variance expressions are similar to the exponential arrival case. The difference for all cases is the appearance of ξ_0 in the expressions in place of ρ. When the interarrival distribution is exponential $\rho = \xi_0$ and all the expressions reduce to the expressions for the exponential case. In view of these similarities, the expression used to estimate the correlation of the X_n and X_{n+1} term will be ascertained from this exponential case with ξ_0 replacing ρ in the proper places. The expression for the exponential case is

$$r(k) = \exp\left(-\frac{k}{\mathrm{var}(L)}\right) \tag{3.43}$$

hence the expression for this case is

$$r(k) = \exp\left[-\frac{(1-\xi_0)^2}{(\xi_0+1)-\rho^2}k\right] \tag{3.44}$$

The problem that arises in applying these formulas is finding ξ_0. To find its value, the Laplace transform of the arrival distribution $a^*(s)$ must be found and the root of the equation

$$\lim_{s \to 0} \underline{a^*}\left[s + (1-X)\mu\right] = X \tag{3.45}$$

has to be computed. Finding the numerical value of ξ_0 is no easy task. The reason for this investigation however is to find easily applied estimations which are close to the real result. An easy method of estimating ξ_0 is required. The approach to estimating ξ_0 will be done as in the case of service time - by comparison to results for mathematically solvable systems.

For standard deviation to mean ratios greater than one ξ_0 will be estimated by the formula for the hyper-exponential case. Morse[17] gives this formula as

$$(1-\xi_0)(2\rho-\xi_0) = \frac{2\rho(1-\rho)}{\frac{\sigma^2}{a^2}+1} \tag{3.46}$$

The root of this equation which is less than one is the estimate of ξ_0. This root is:

$$\xi_0 = \rho + \tfrac{1}{2} - \sqrt{\frac{(2\rho+1)^2}{4} - 2\rho - \frac{2\rho(1-\rho)}{\frac{\sigma^2}{a^2}+1}} \tag{3.47}$$

When the standard deviation to mean ratio is less than one, a comparison to the Erlang case will be made. The expression for ξ_0 changes as the variability of the Erlang distribution changes. This will necessitate a comparison of two extreme cases. The first is exponential where $\xi_0 = \rho$. The second is regular arrivals, where the variability of the arrivals is zero. Here the value of ξ_0 is the solution to

$$\xi_0 = 1 - \rho \ln\left(\frac{1}{\xi_0}\right) \tag{3.48}$$

The values of ξ_0 between variability ratio one and zero will be found by linear interpolation.

The case where both the service and arrival times are non-exponential is difficult to solve mathematically. To estimate the properties of this system a simpler heuristic approach will be used.

Since the service and arrival time affect the magnitudes of the properties in the same direction for equal coefficients of variance, the coefficients of variance will be multiplied together and this number treated as a coefficient of arrival variation. This is a rather crude method of estimation, but it satisfies the criterion of simplicity needed for advanced estimating.

IV THE COMPUTER SIMULATION

Several approximations were made to obtain the formulas derived in the last chapter. In order to verify the validity of these approximations, simulations of the models were made. The objective of these simulations is to verify the numerical values of the equations for variance, correlation, and sample variance. The simulation technique used is an adaptation of the variable time increment model of Naylor, et.al.[18]. The line length is sampled at regular intervals and the raw statistics recorded. The condition of the system was updated only when an arrival, departure, or beginning of service took place.

The generation of service and arrival time are performed by subroutines so that the distributions can be changed easily. Raw statistics are gathered at multiples of the average service interval. At intervals controlled by a data card, the statistics are computed in their final form and printed out. This allows observation of the properties of the system as the simulation progresses. A flow diagram of operations is given in Appendix I. The notation used is as follows.

MAX - The sample count at which statistics will be printed out.

IMX - Maximum number of periods for which correlations are computed.

NWL - Number in waiting line.

AT - inter-arrival time.

SUMAT- Clock time of next arrival.

ST - Service time.

TWT - Total waiting time.

ISTA - Service Station indicator (0 if empty; 1 if occupied)

CLOCK - Cumulative time.

KOUNT - Departure count.

CHK - Time at which next sample is to be taken.

N - Path indicator.

ITERM - Sample count.

TIDL - Total idle time.

CVA - Coefficient of variation of arrivals = $\lambda\sigma_a$

CVS - Coefficient of variation of service time = $\mu\sigma_s$.

The portion of program from the start to C is for setting initial conditions and controlling run length. The section from C to E is the queuing system model. The service and arrival times are generated in this section and the state of the system is determined at appropriate times. From E to F, the statistics are computed and printed out. Raw statistics are collected whenever the simulator clock exceeds the sample alarm CHK. Final statistics are computed whenever the sample count reaches MAX. A new MAX is then read in and the simulation continues or terminates depending upon MAX. When a new problem is obtained, the simulator is run for a specified period of time (the first MAX) and then the statistics are initialized. This initial period is required for the model to reach steady state. The condition of the queuing system at this point is the starting state for the run. The length of the transient periods were determined by use of equation (3.26). For the transients to become negligible compared to the steady state, a period of 3Z was allowed. The values of transient time (in sample periods) used are

ρ	Time
.7	125
.8	250
.9	1500
.95	5000

These are rounded values of 3Z for the respective ρ's.

The statistics computed by the program are

Average line length

$$L = \sum_{i=1}^{N} \frac{X_i}{N} \tag{4.1}$$

Covariances of X_i and X_{i-k}

$$R(k) = \sum_{L=1}^{N-k} \frac{(X_L-L)(X_{i-k}-L)}{N-k} \tag{4.2}$$

Correlation coefficient of X_i and X_{i-k}

$$r(k) = \frac{R(k)}{R(0)}$$

The sample variance

$$\sigma_s^2 = \frac{1}{N}\left[R(0) + 2\sum_{k=1}^{IMX}\left(1 - \frac{k}{IMX}\right) r(k)\right] \tag{4.3}$$

Two additional properties of the system were recorded as checks on the model

Percent idle time

$$PCI = \frac{TIDL}{CLOCK} \tag{4.4}$$

Average waiting time in the line and in service

$$AVGWT = \frac{TWT}{KOUNT} \tag{4.5}$$

To test the results two distributions were investigated -- the Weibull and the normal. The Weibull was selected because it is an excellent imitator of most time random variables[1,2] and because it is a two paramater distribution which allows it to be transformed into a wide range of shapes. The density function for the Weibull distribution is[7]

$$f(t) = \alpha\beta t^{\beta-1} \exp(-\alpha t^{\beta}) \tag{4.6}$$

Its mean and variance are

$$\mu = (\alpha)^{-1/\beta} \, \Gamma(1 + 1/\beta) \tag{4.7}$$

$$\sigma^2 = (\alpha)^{-2/\beta} \left\{ \Gamma(1 + 2/\beta) - \left[\Gamma(1 + 1/\beta)\right]^2 \right\} \tag{4.8}$$

where $\Gamma(X)$ is the gamma function of X.

The normal distribution was selected because of its applications to queue theory[15] and because using a second distribution allows a check on how sensitive the results are to distributions. The normal density is

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-\mu)^2/2\sigma^2} \tag{4.9}$$

Its mean and variance are μ and σ^2 respectively.

Four values of ρ were used in each group of simulations. The values chosen were .7, .8, .9, and .95. Values were chosen closer to one than zero because these produce the largest sample sizes and are of the most interest.

Sample sizes for the experiment were selected as 8000 samples for ρ of .7 and .8 and 16,000 samples for ρ of .9 and .95. These lengths were adequete to obtain accurate results in most cases. The predetermined sample sizes allowed the sample variance estimates to be computed by the same procedure for all cases and allowed computer run times to be stated in advance (a requirement of the computing center).

In all, one hundred simulations were run for values of ρ equal to 0.7, 0.8, 0.9, and 0.95. Four simulations with exponential service and arrival times were run to establish the validity of the program. Twenty eight runs were made using Weibull distributed inter-arrivals with β parameters of .3, .5, .7, 1.5, 2.0, 2.5, and 3.0. Twenty-eight more simulations were done using Weibull distributed service times with the same parameters as for the inter-arrival times. In order to check a different type of distribution, normally distributed service and arrival times were simulated. Mean to standard deviation ratios of 0.3, 0.5 and 0.7 were used to simulate N/M/1 and M/N/1 systems. Finally, the case where both service and arrivals were Weibull distributed was investigated. Combinations of high and low variances were taken. For the values of ρ given above, runs were made with β parameter pairs of (.5, .5), (.5,2), (2,.5), and (2,2), where (β_S, β_A) are service and arrival distribution parameters respectively.

The program was written in Fortran IV language for the IBM 360-50 computer. A 16,000 sample simulation took approximately five minutes running time. Memory requirement, including subroutines, is nearly 6500 bytes.

V DISCUSSION OF RESULTS

The first simulation run was for a M/M/1 system. The results of this run are shown in Appendix II. This run was made primarily for the purpose of testing the validity of the program. The values of the average line length, variance of the line length, sample variance, and correlation coefficient obtained from simulation are plotted for values of ρ of .7, .8, .9, and .95. The most influential factor affecting the estimate of sample accuracy is the value of the variance of the line length. As is shown, the line variance suffered the greatest variation from predicted results. This is not unexpected for the expression for variance is more sensitive to ρ than the other expressions. The values of the line length, correlation coefficient, and sample variance are close to the mathematically known values and is reassurance that the model works correctly.

The second group of simulations were performed for the Weibull arrival case. First values of β which made the CVA ratio smaller than one were chosen. These were β = 1.5, 2.0, 2.5, and 3.0 which gave CVA's of .68, .52, .43, and .36 respectively. The results are shown in Appendix III. The values of line length were less than for the corresponding exponential case as predicted by the line length formulas. The approximate estimates of line length shown in Figure 10 predict the simulated values of line length are within the sampling accuracy of this experiment. Variance of the line length is erratic and the differences between the various runs is not as apparent as for line length. This is particularly true for the ρ = 0.9 and 0.95

cases where the lower CVA ratios sometimes had greater line length variance than the higher CVA ratios. The same pattern is observed for the correlation coefficients and sample variances where the effect of CVA's for $\rho = 0.9$ and 0.95 was lost in the experimental error. Comparison of the calculated values and simulated values indicates that the agreement is as close as one would require for advanced planning of experiments.

The third set of simulations was for Weibull arrivals and CVA ratios greater than one. Values of the β parameter in the Weibull distribution were .3, .5 and .7 giving CVA's of 1.46, 2.24, and 5.4 respectively. The results are in Appendix IV. The values of line length, variance of line length, correlation coefficient, and sample variance are all substantially increased over the exponential case as expected. For a given value of ρ, the numerical value of the system properties increased with increasing CVA. The average line lengths predicted by equation (3.41) lie within the 95% confidence intervals obtained by simulation. The line length variances agree within a factor of 2 with a tendency for the predicted values to underestimate the simulation results.

The next group of simulations was for normally distributed inter-arrival times and exponential service times. The results are shown in Appendix V. Calculated values of the system properties tended to be slightly higher than the actual results, however, the calculated values still predict the results without large error.

In Appendices VI, VII, and IX are the results for non-exponential service times. Appendix VI is for Weibull service times and CVS of less than one. Appendix VII is for CVS >1, and Appendix IX for normal service times. The CVS ratios were the same as the corresponding CVA ratios in the previous set of simulations. A comparison of the results with the predictions of Chapter 3 shown in Figures 25, 30, and 35 reveals that the predictions are poor for r(k), var(L), and σ_s. The estimates of var(L) are not as sensitive to CVS as the simulation indicates that they should be. The results are quite similar to the non-exponential arrival results. The tendency is for the numerical values to be smaller for the arrival time case with corresponding ρ and CVA, but by small amounts. The significant observation is that the predictions for arrival times predict the non-exponential service distribution results with much less error than the predictions in Figures 25, 30, and 35.

The last group of simulations were for the case where both the arrival and service distribution were non-exponential. Calculated values in Figure 40 and experimental results in Figures 36, 37, 38 and 39 were not very close. For the case where both CVA and CVS were less than one, the calculated values overestimated the results. The two cases where one coefficient of variation was greater than one and one less than one produced approximately similar results. However the calculated values underestimated the magnitude of the system properties. The final case where both CVA and CVS were greater than one was overestimated by the calculated predictions.

During the simulation runs, the properties plotted in the Appendices were periodically printed out so that transient biases, if any, could be observed. No transient growths were observed by eye except for cases where CVA and CVS were greater than one and $\rho = 0.95$. In these cases the growth in the properties leveled off before the end of the simulation.

VI CONCLUSIONS

(1) An estimate of the sample variance in simulation of an M/M/1 queuing system may be obtained by using the formulas

$$\mathrm{Var}(L) = \frac{\rho}{(1-\rho)^2} \tag{6.1}$$

$$r(k) = \exp\left(-\frac{(1-\rho)^2}{\rho} k\right) \tag{6.2}$$

where $k=\mu t$ is the number of average service intervals between samples.

$$\hat{\sigma}_s^2 = \text{sample variance} = \frac{\mathrm{Var}(L)}{N}\left[1+2\sum_{i=1}^{IMX}\left(1-\frac{i}{IMX}\right)r^i(k)\right] \tag{6.3}$$

where N is the number of samples.

If an accuracy specification for line length is given in the following form

$$p_r\left(\left|\frac{L-\hat{L}}{L}\right| < \alpha\right) \geq \beta \tag{6.4}$$

then a confidence interval may be obtained from the equivalent expression

$$p_r\left[L(1-\alpha) < \hat{L} < L(1+\alpha)\right] \geq \beta \tag{6.5}$$

In terms of σ_s this expression is

$$p_r\left[L-T_s\sigma_s < \hat{L} < L+T_s\sigma_s\right] \geq \beta \tag{6.6}$$

where T_s is obtained from the tables of the t-distribution for a given α. The degrees of freedom will depend on N and as an initial approximation the normal distribution may be used in place of the t-distribution.

N may be computed by

$$N = \frac{Var(L)}{T_s^{\,2}\,\sigma_s^{\,2}} \left[1 + 2 \sum_{L=1}^{IMX} \left(1 - \frac{i}{IMX}\right) r^{i}(k) \right] \qquad (6.7)$$

The degrees of freedom for t-distribution are[6]

$$DOF = \frac{1.5N}{IMX} \qquad (6.8)$$

If this value is thirty or more, the normal approximation is valid, but if not, more successive approximations of N will be required.

(2) The results of the simulations indicate that when either the service or interarrival time distributions are non-exponential, the sample lengths required depend upon both and the coefficient of variations CVA and CVS. Equation 6.7 still approximates the results, but new values of Var(L) and r(k) are required.

(3) For non-exponential arrivals, the methods given in Chapter 3 for computing ξ_0, Var(L), and r(k) agree with simulated results within accuracy required for purposes of advanced sample size estimates.

(4) For non-exponential service times, the estimates of Var(L) and r(k) given in (3.30) and (3.32) Chapter 3 are not accurate. However, the values obtained for this case do not differ substantially from the values obtained for non-exponential arrivals with CVS substituted for CVA.

(5) The prediction of the properties for the case where both the arrival and service distribution by multiplying CVA and CVS and using the result as though it were a new CVA does not predict

the results as accurately as in the previous cases. If this technique is used, the experimenter must be prepared to accept somewhat larger errors than in the previous cases.

(6) To the estimate of sample size N, the transient time must be added. Estimation of this time by use of equation (3.26) appears adequate except when CVA or CVS are larger than one and ρ large (.95).

The methods given for the calculation are somewhat involved. To arrive at a value of sample size for an non-exponential queue requires calculation of ξ_0, Var(L), and r(k) and N. In an experiment where a large number of separate simulations are required, the effort needed for estimation of all sample sizes would be enormous. Evaluation of the best and worst cases will give the experimenter an idea of the order of magnitude of sampling required for the total investigation and save considerable computing.

<u>Recommendations for Further Study</u>

The results obtained in this study were for the simplest type of queuing system -- the single serve with no priorities. There are a multitude of queuing systems not covered by this category and simulation studies of these other types are common. Any of these systems might be a candidate for a study of the kind presented here. Probably the most fruitful area for further study of sample size would be in multiple stage and multiple channel systems. The results obtained in this study might be helpful in the investigation of these systems. There are also many features such as bulk queuing, priority systems, and limited lines which are not covered here. These are all common queuing phenomena worth investigating.

Other properties of the queuing system besides line length are also of interest. Some of these, such as waiting time and queue length are closely related to the line length as would be their sampling properties. Others, such as maximum queue length, would require a completely different sampling experiment.

APPENDIX I

SIMULATION FLOWCHART

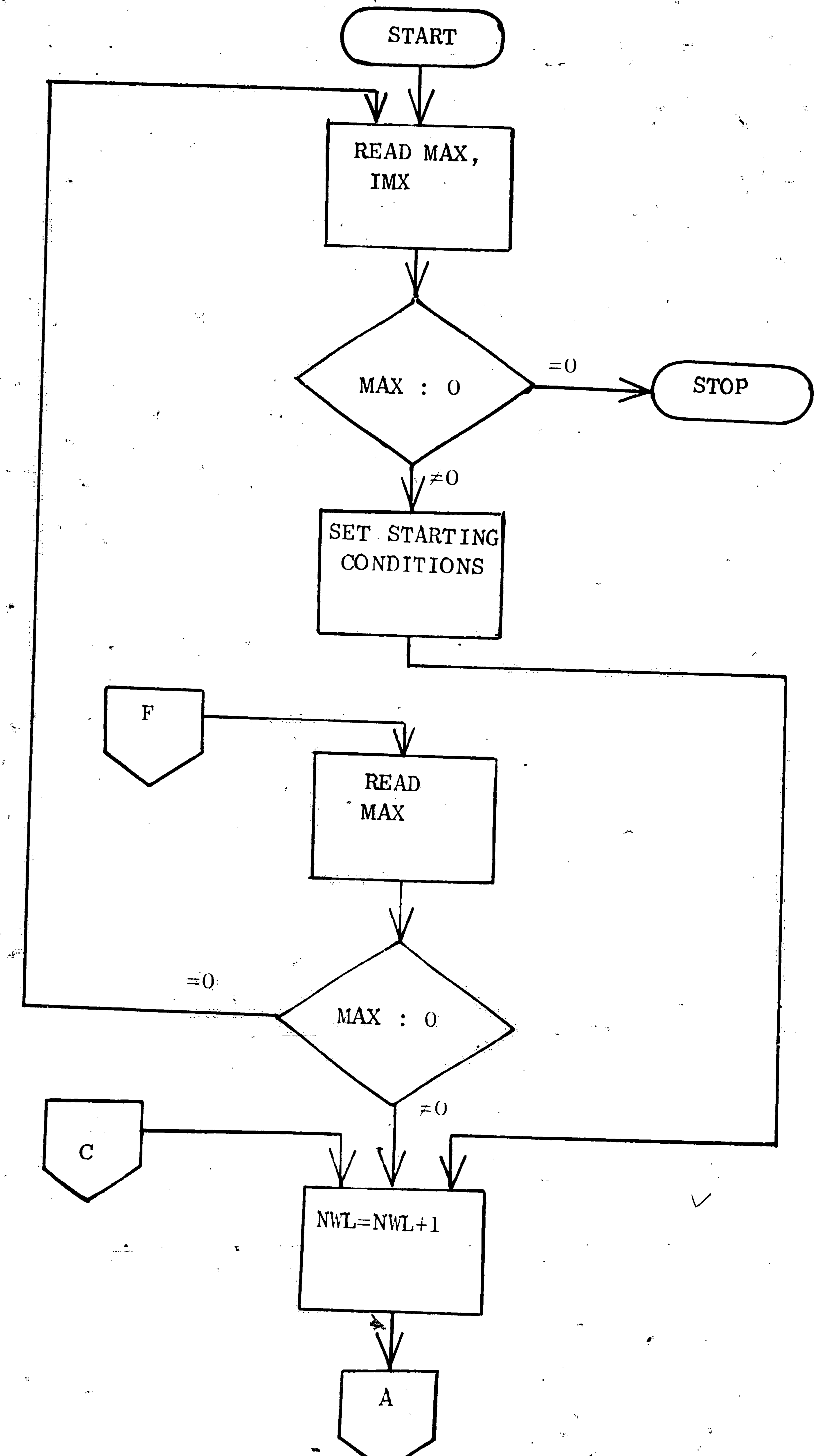

Figure 1. Simulation Flowchart

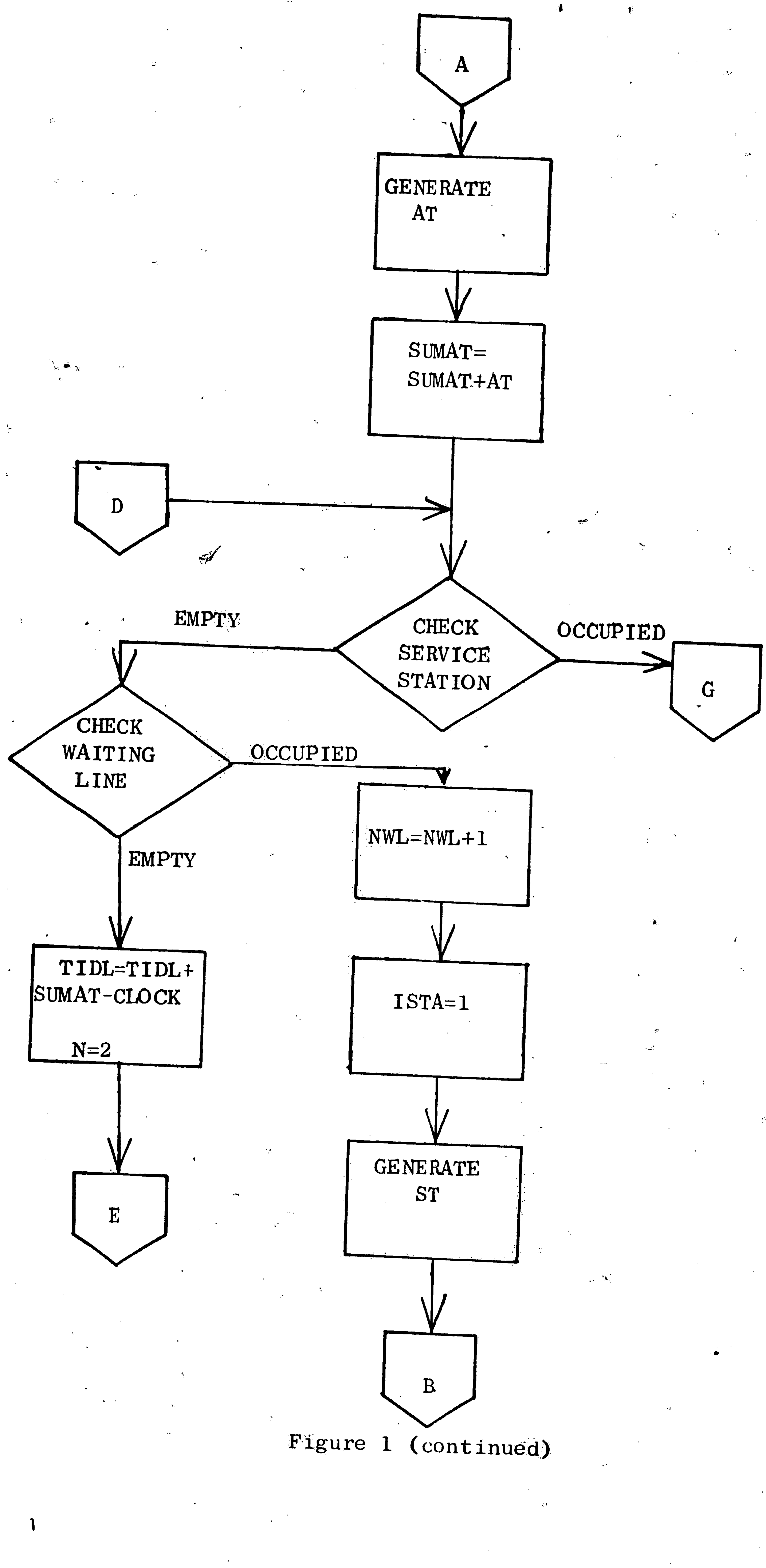

Figure 1 (continued)

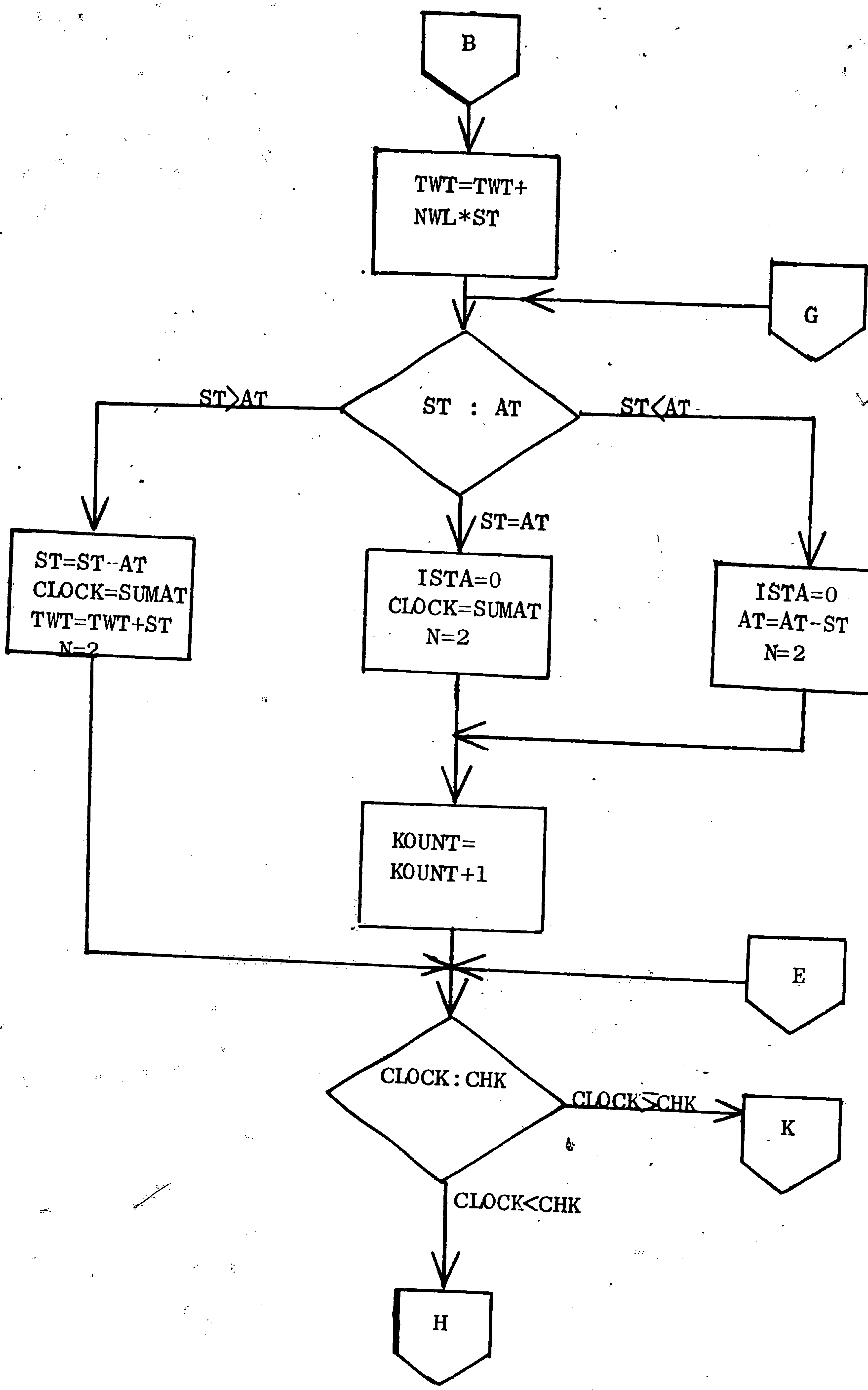

Figure 1. (continued)

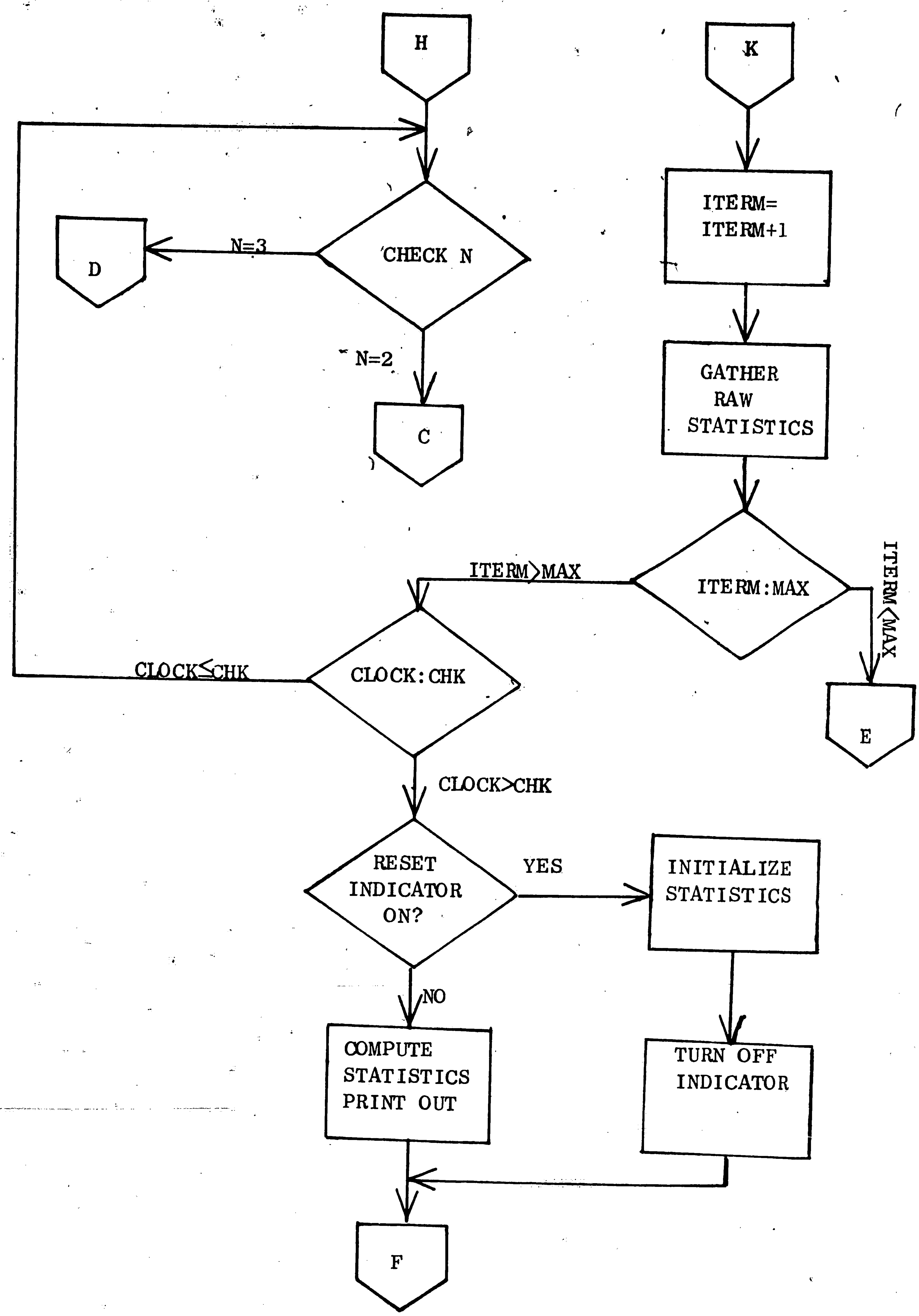

Figure 1. (concluded)

APPENDIX II

RESULTS OF

M/M/1 SIMULATIONS

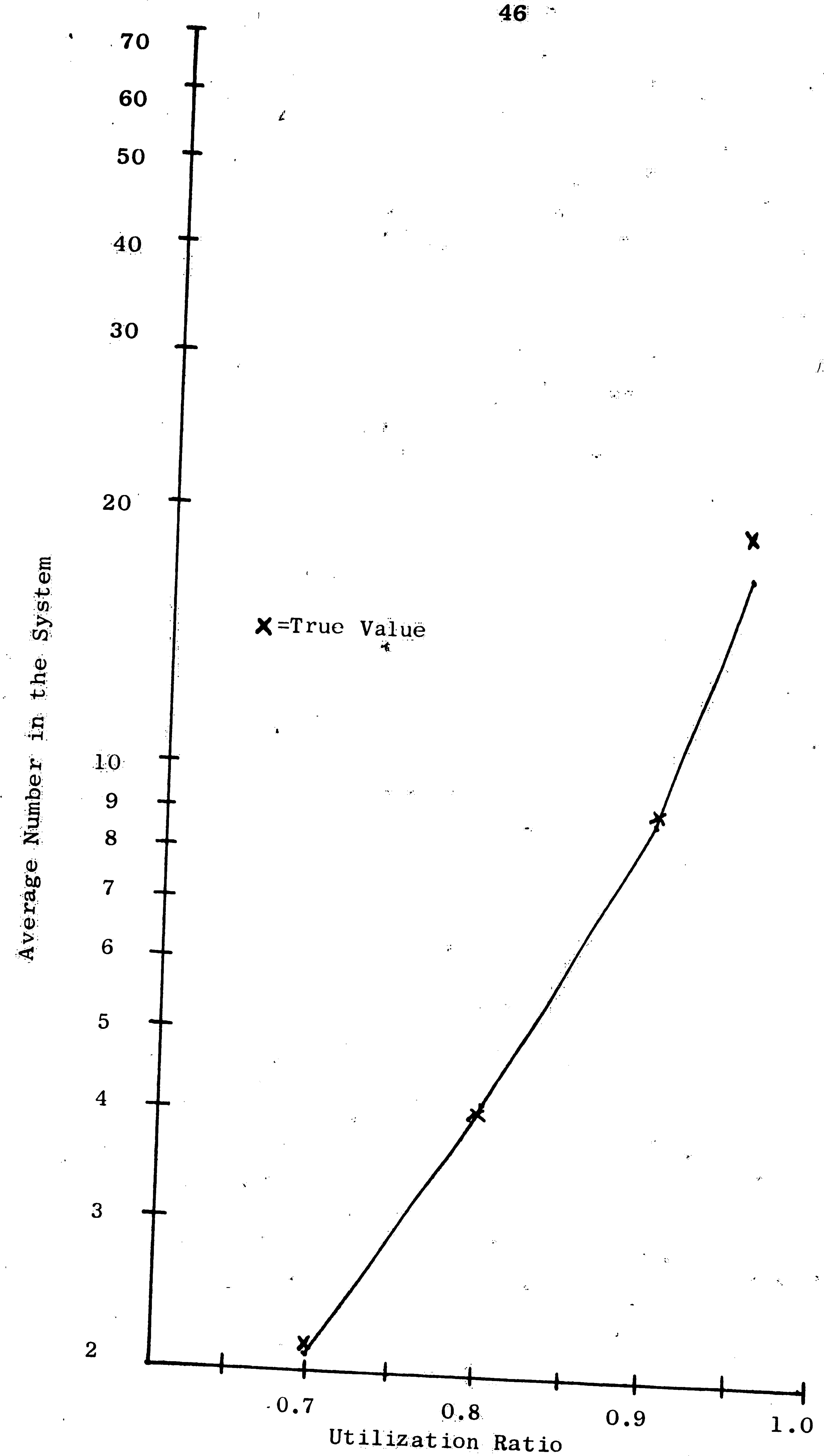

Figure 2 Line Length of M/M/1 System

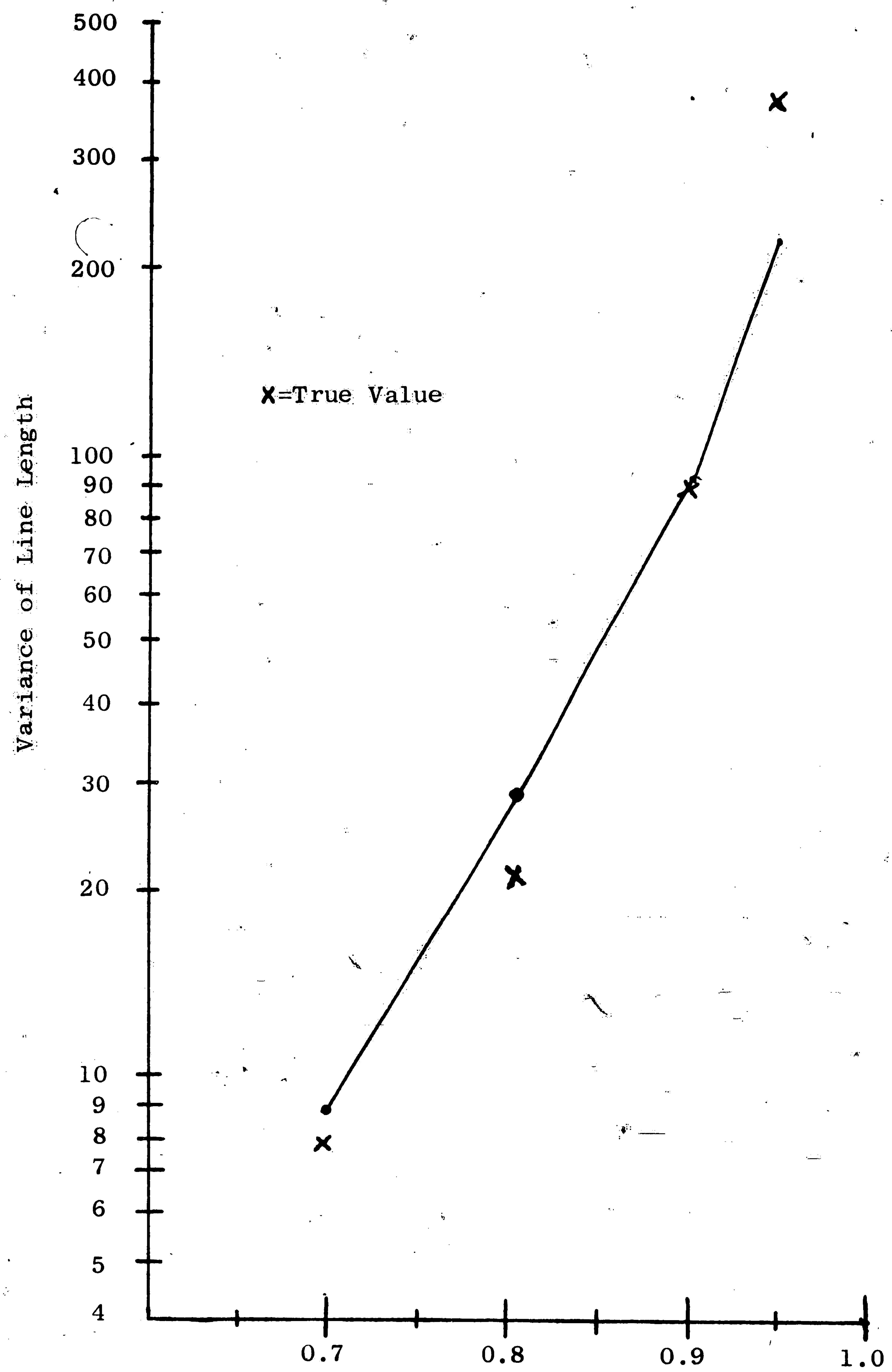

Figure 3 Variance of Line Length for M/M/1 System

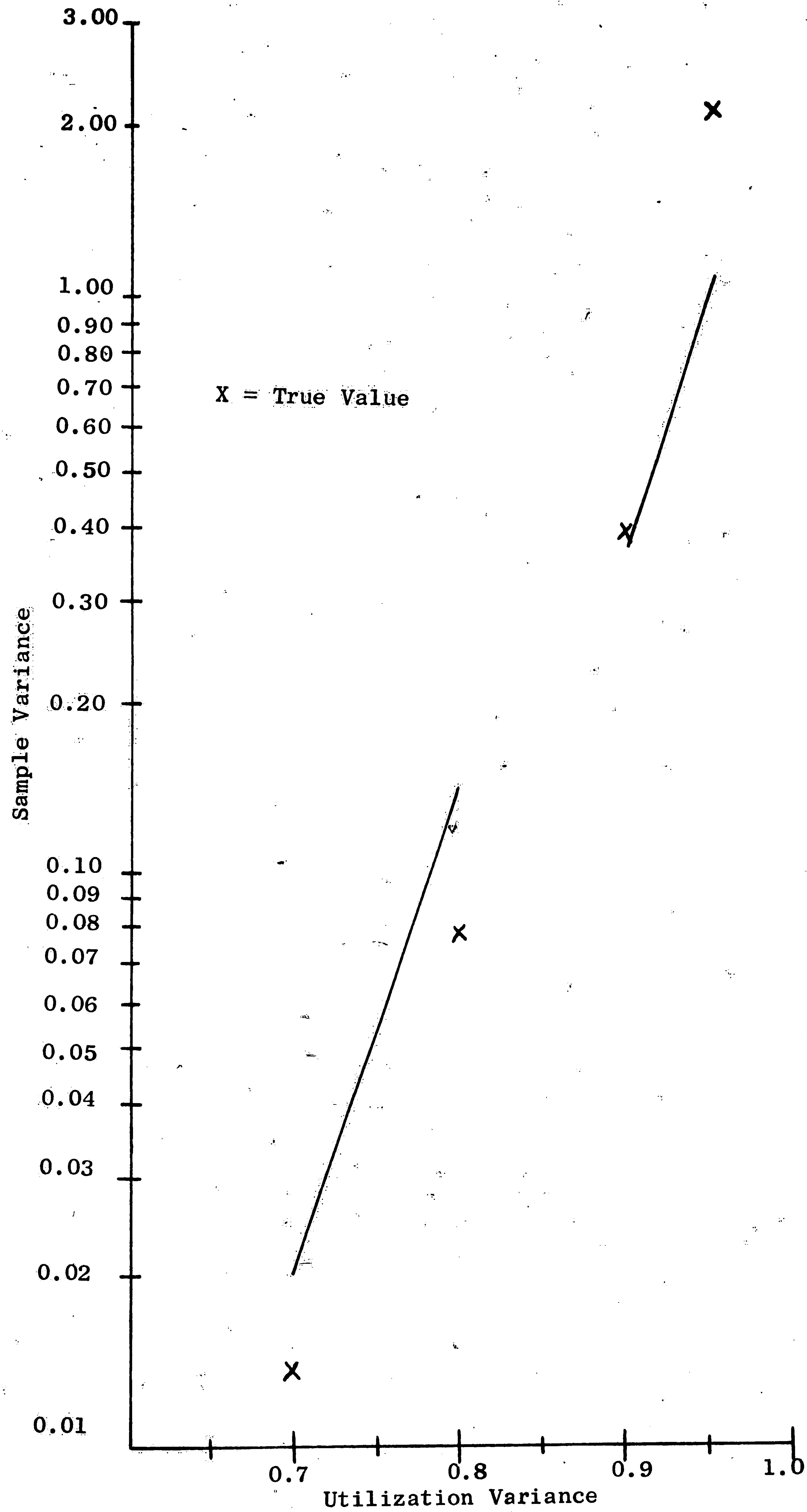

Figure 4 Sample Variance of M/M/1 System

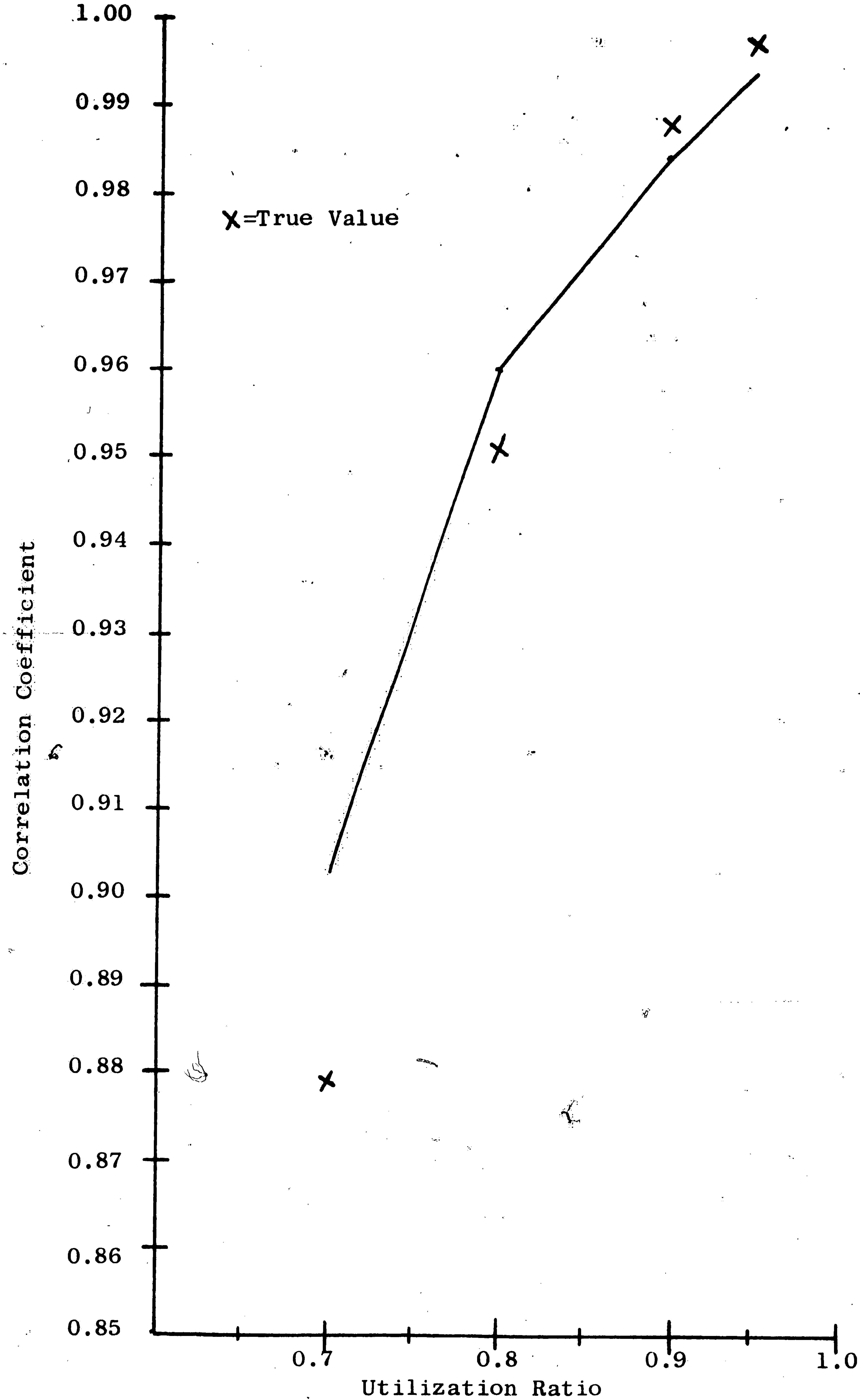

Figure 5 Correlation Coefficient for M/M/1 System

APPENDIX III

RESULTS OF

W/M/1 SIMULATIONS CVA<1

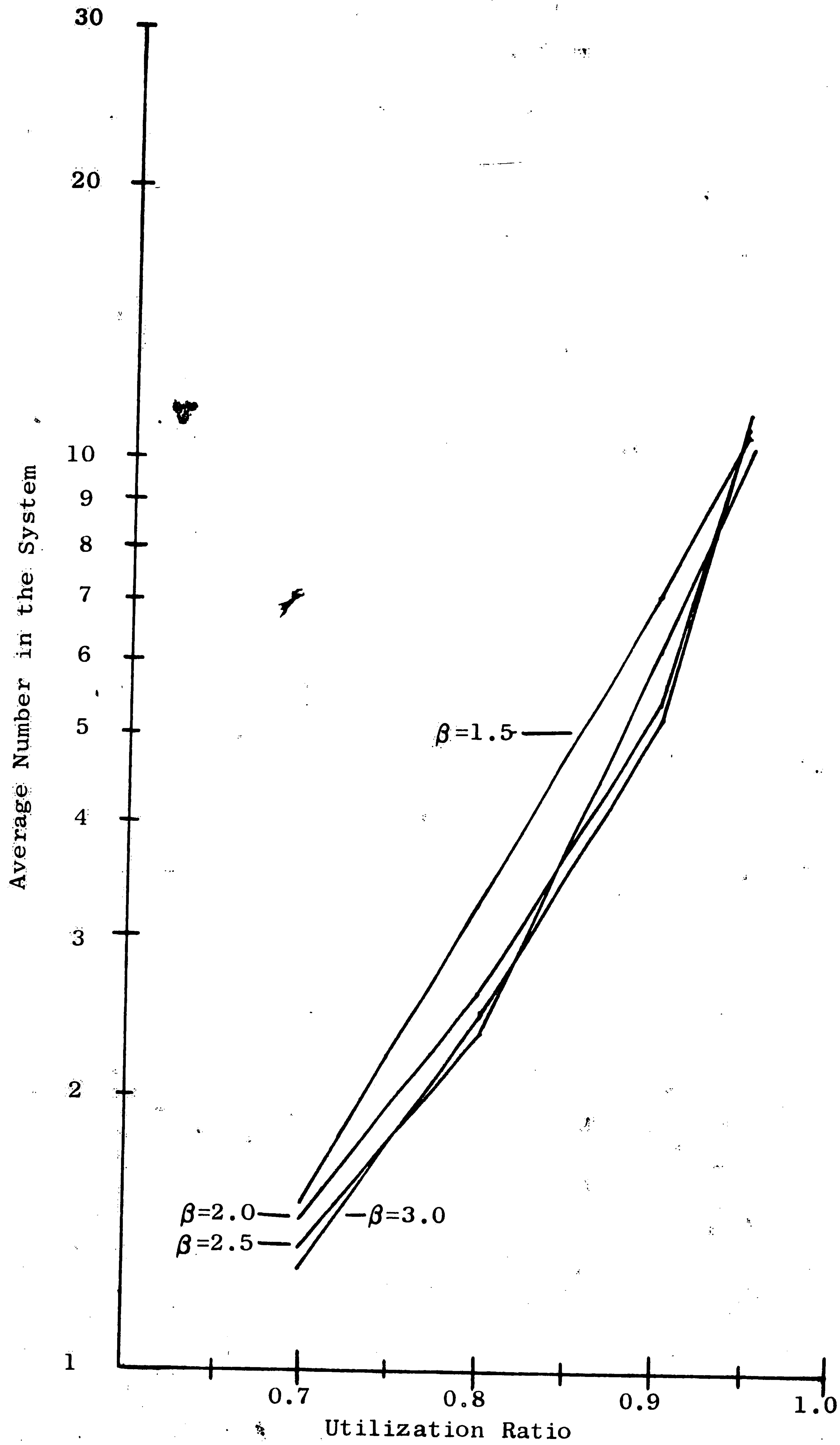

Figure 6 Line Length for W/M/1 System CVA<1

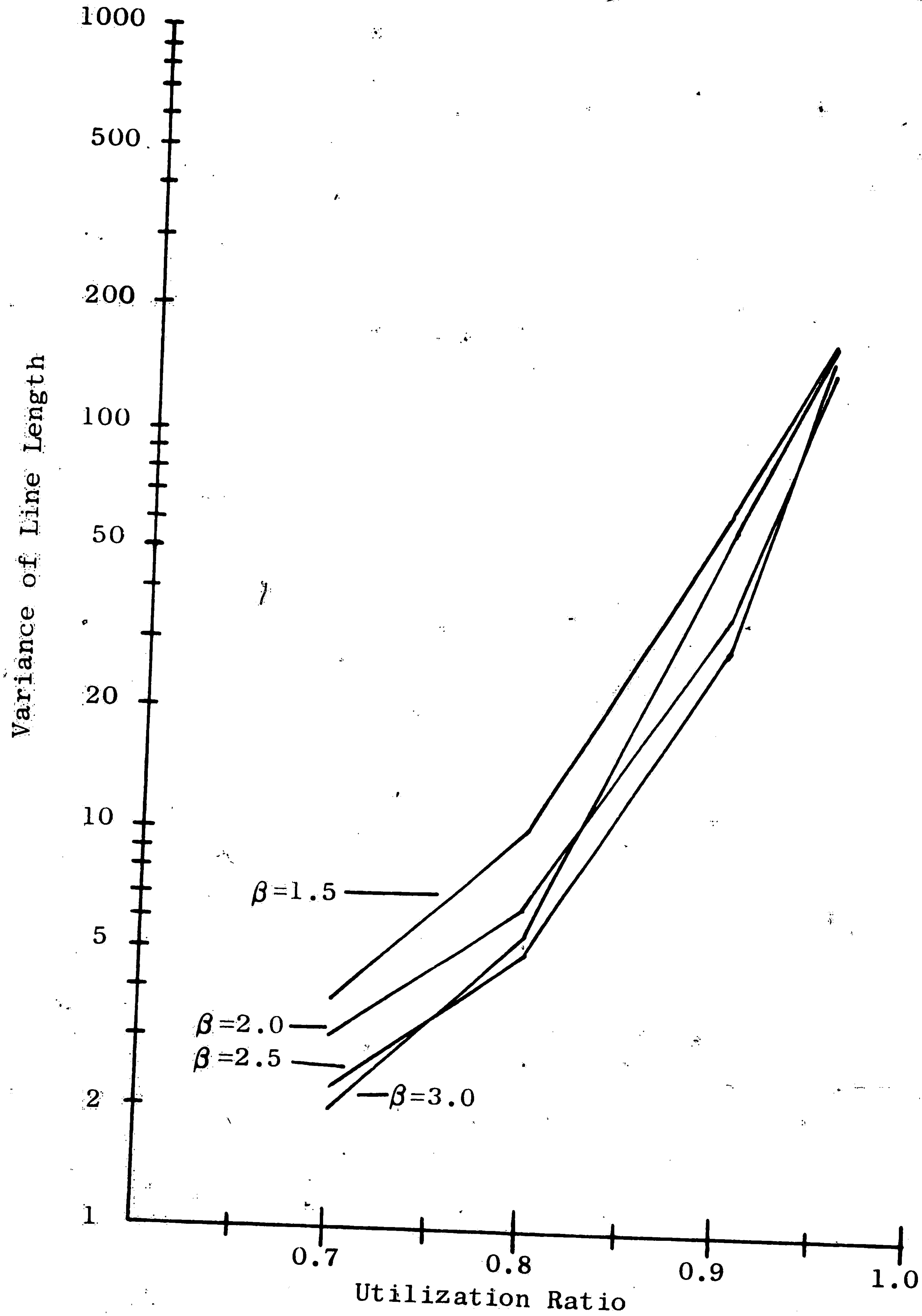

Figure 7 Variance of Line Line Length for W/M/1 system CVA<1

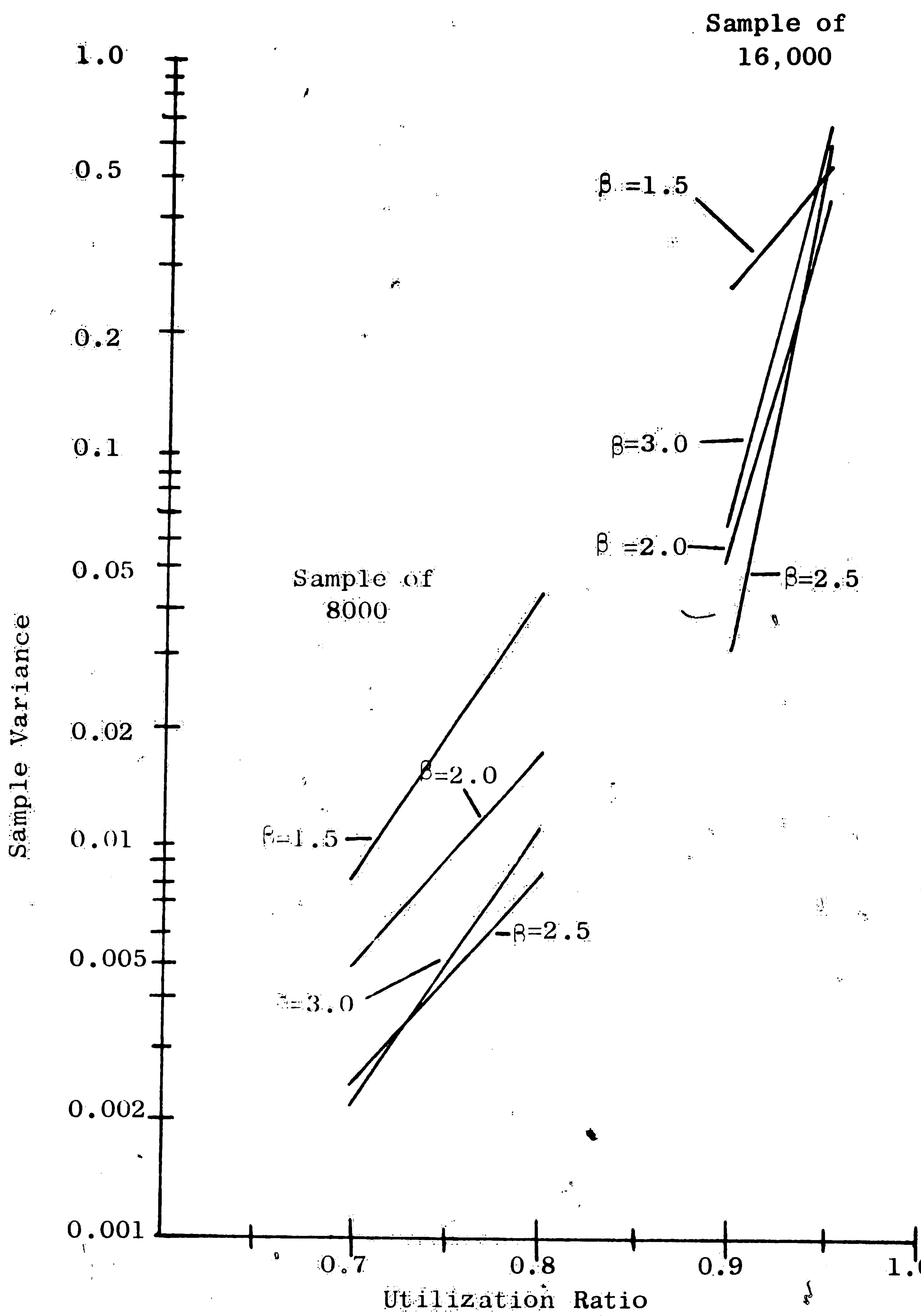

Figure 8 Sample Variance for W/M/1 System CVA<1

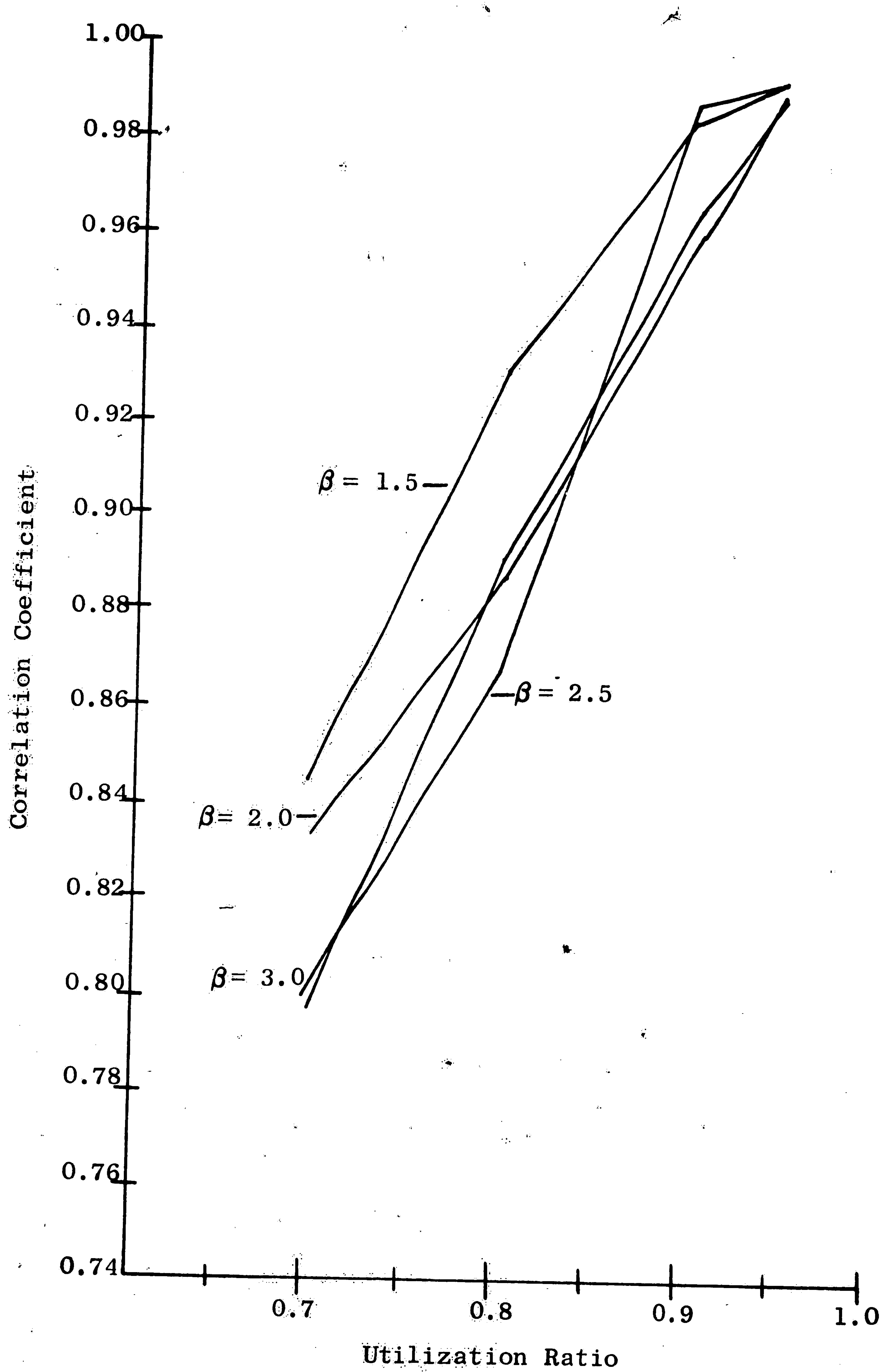

Figure 9 Correlation Coefficient for W/M/1 System CVA < 1

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		β=1.5		
0.7	1.86	4.60	0.005	0.805
0.8	3.13	11.62	0.029	0.917
0.9	6.92	51.64	0.186	0.980
0.95	14.74	225.42	1.221	0.995
		β=2.0		
0.7	1.70	3.67	0.003	0.762
0.8	2.83	9.24	0.019	0.897
0.9	6.22	41.10	0.132	0.975
0.95	13.29	181.95	0.954	0.994
		β=2.5		
0.7	1.61	3.23	0.002	0.733
0.8	2.68	8.11	0.015	0.884
0.9	5.86	36.15	0.108	0.972
0.95	12.54	161.32	0.828	0.993
		β=3.0		
0.7	1.57	3.00	0.002	0.717
0.8	2.60	7.56	0.013	0.867
0.9	5.67	33.74	0.096	0.970
0.95	12.16	151.20	0.767	0.993

Figure 10

Calculated Properties for Weibull Arrivals

CVA<1

APPENDIX IV

RESULTS OF

W/M/1 SIMULATIONS CVA>1

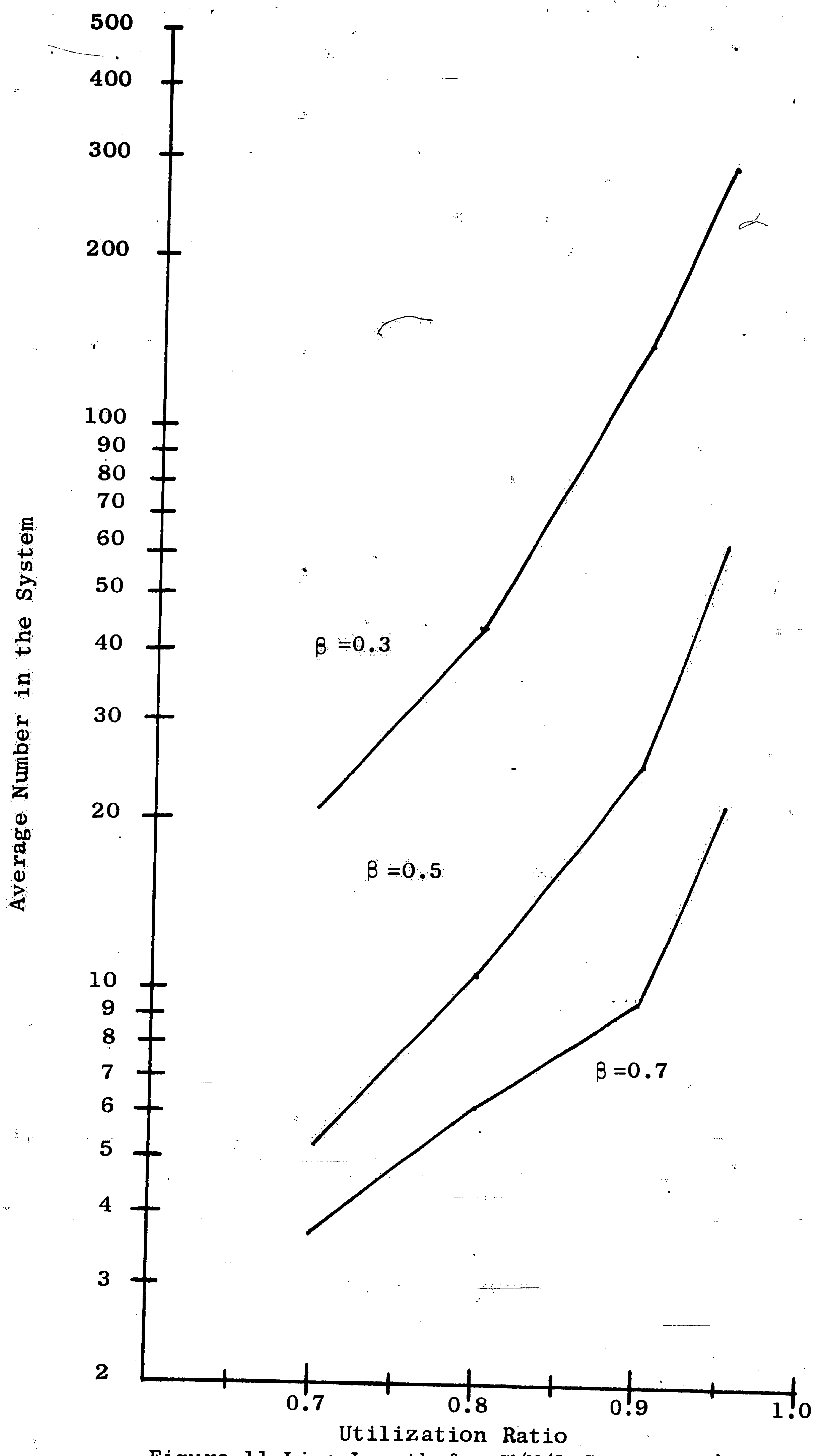

Figure 11 Line Length for W/M/1 System CVA>1

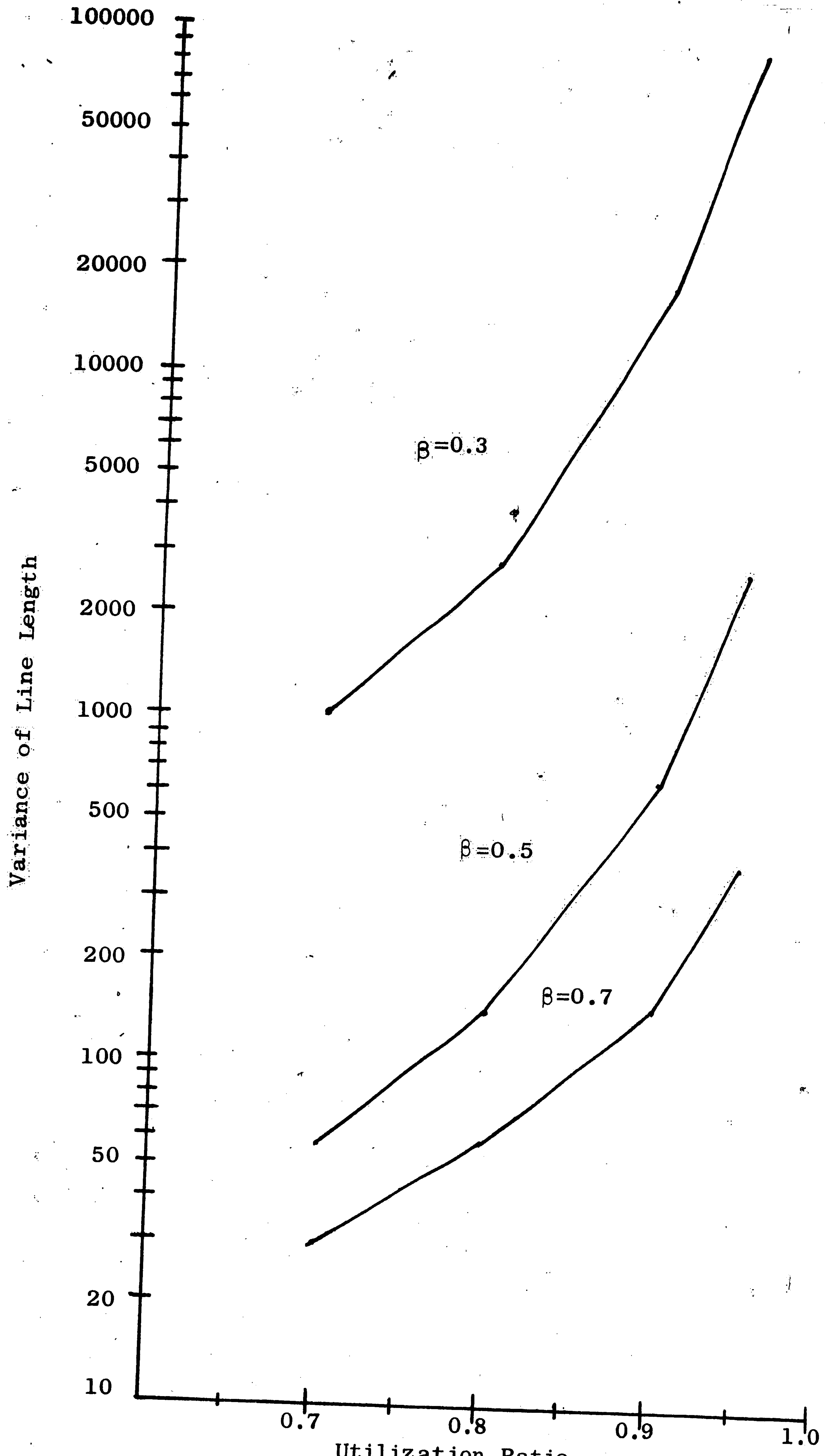

Figure 12 Variance of Line Length for W/M/1 System CVA>1

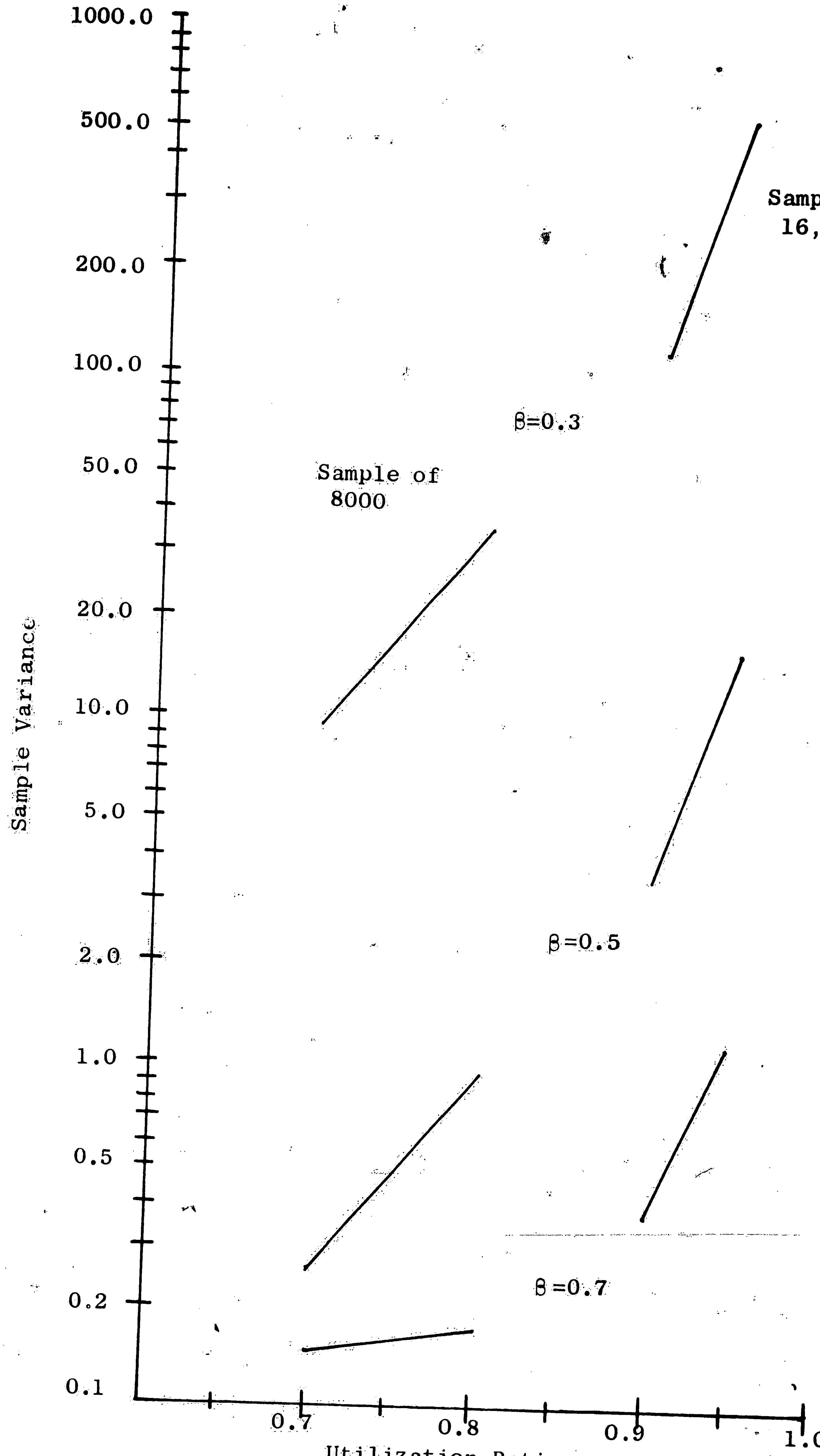

Figure 13 Sample Variance of W/M/1 System CVA>1

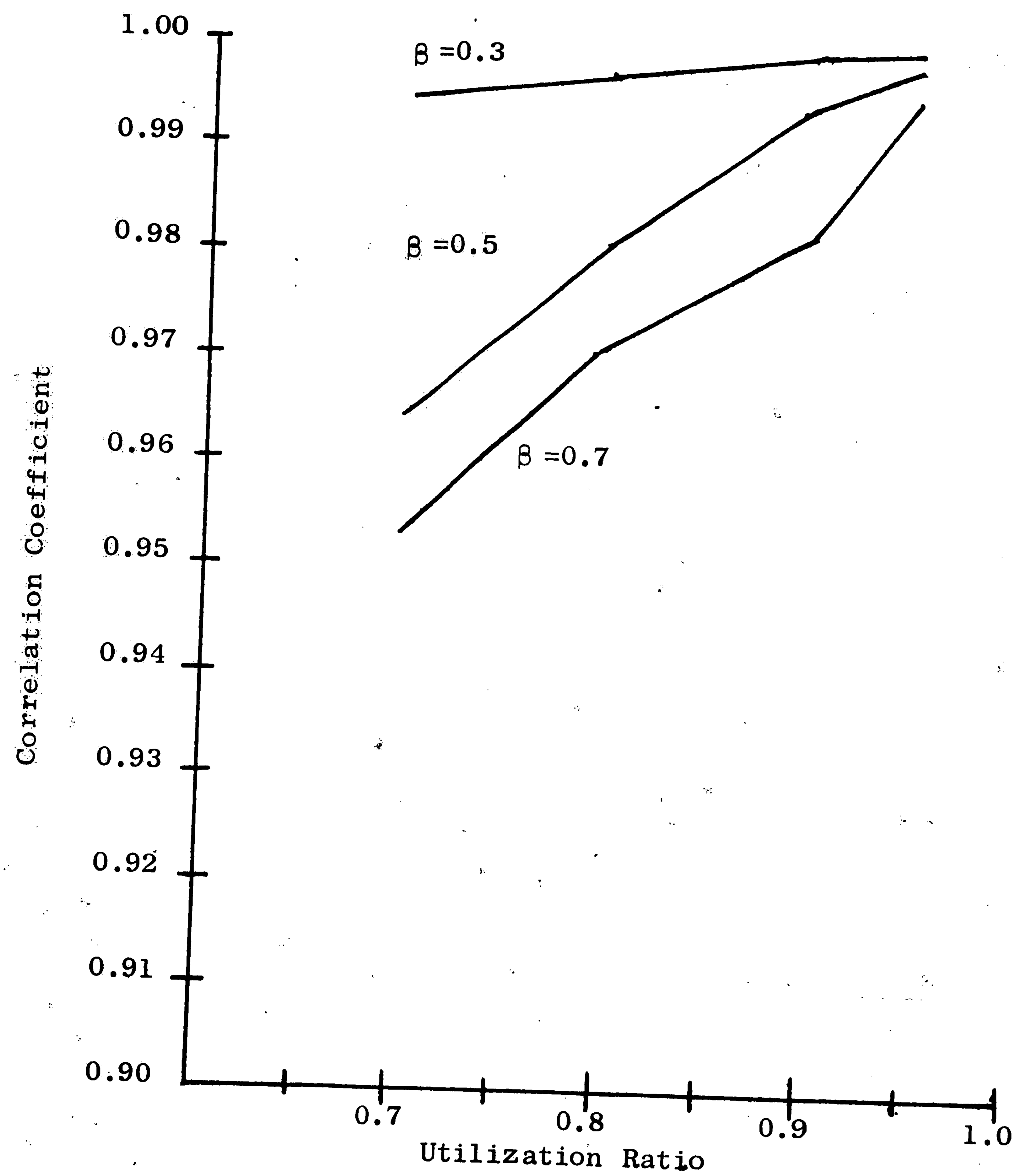

Figure 14 Correlation Coefficient for W/M/1 System CVA>1

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		$\beta=0.3$		
0.7	21.78	859.38	10.33	0.9988
0.8	46.66	3219.22	39.82	0.9996
0.9	122.09	18096.29	112.89	0.9999
0.95	273.29	82276.81	514.01	0.9999+
		$\beta=0.5$		
0.7	5.31	47.17	0.325	0.979
0.8	10.17	145.23	1.461	0.993
0.9	25.07	743.51	4.445	0.998
0.95	55.03	3292.96	20.374	0.999
		$\beta=0.7$		
0.7	3.22	16.11	0.0544	0.939
0.8	5.79	44.52	0.2983	0.977
0.9	13.59	212.26	1.1407	0.995
0.95	29.26	917.48	5.3145	0.999

Figure 15

Calculated Properties for Weibull Arrivals

CVA>1

APPENDIX V

RESULTS OF

N/M/1 SIMULATIONS

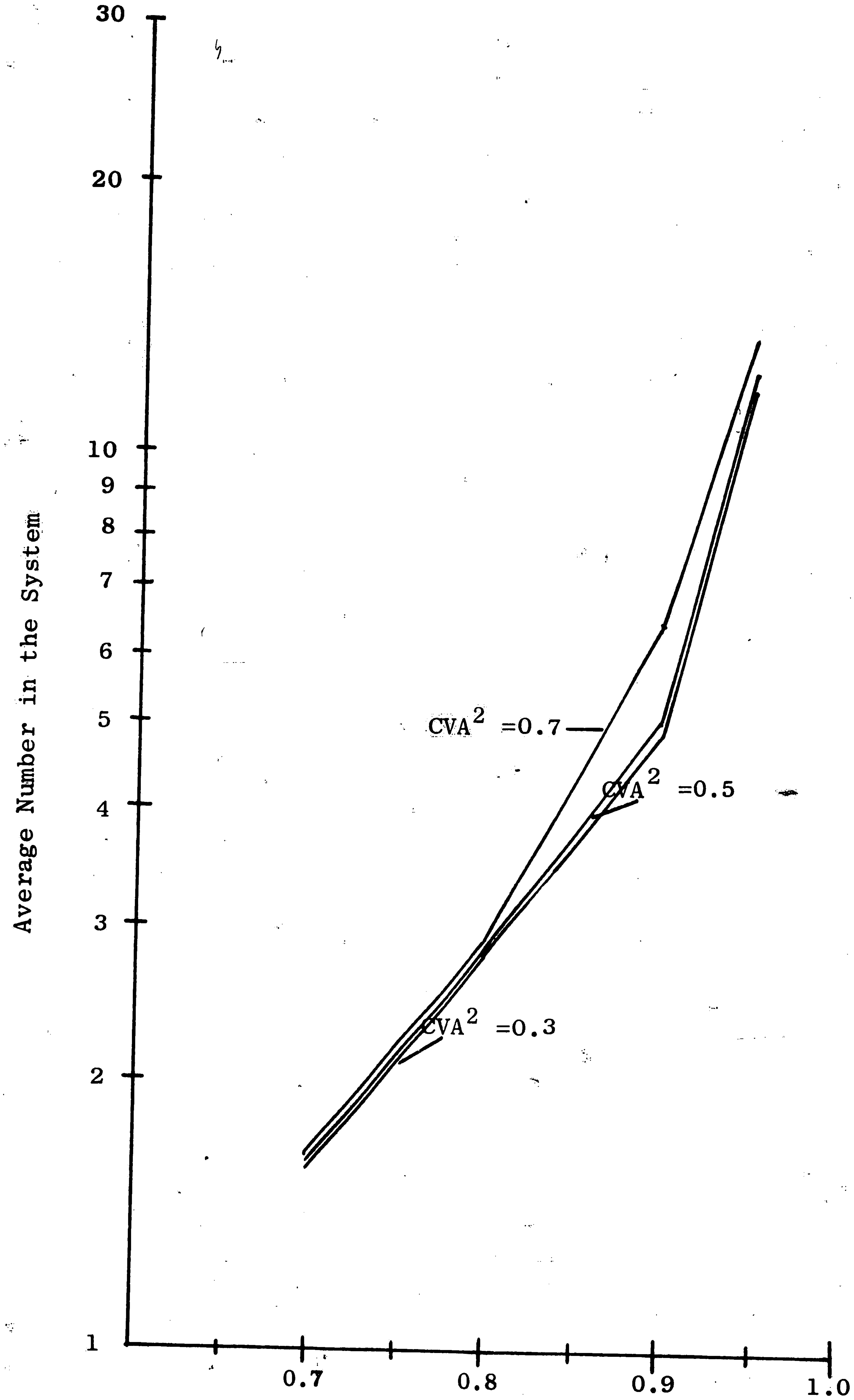

Figure 16 Line Length for N/M/1 System

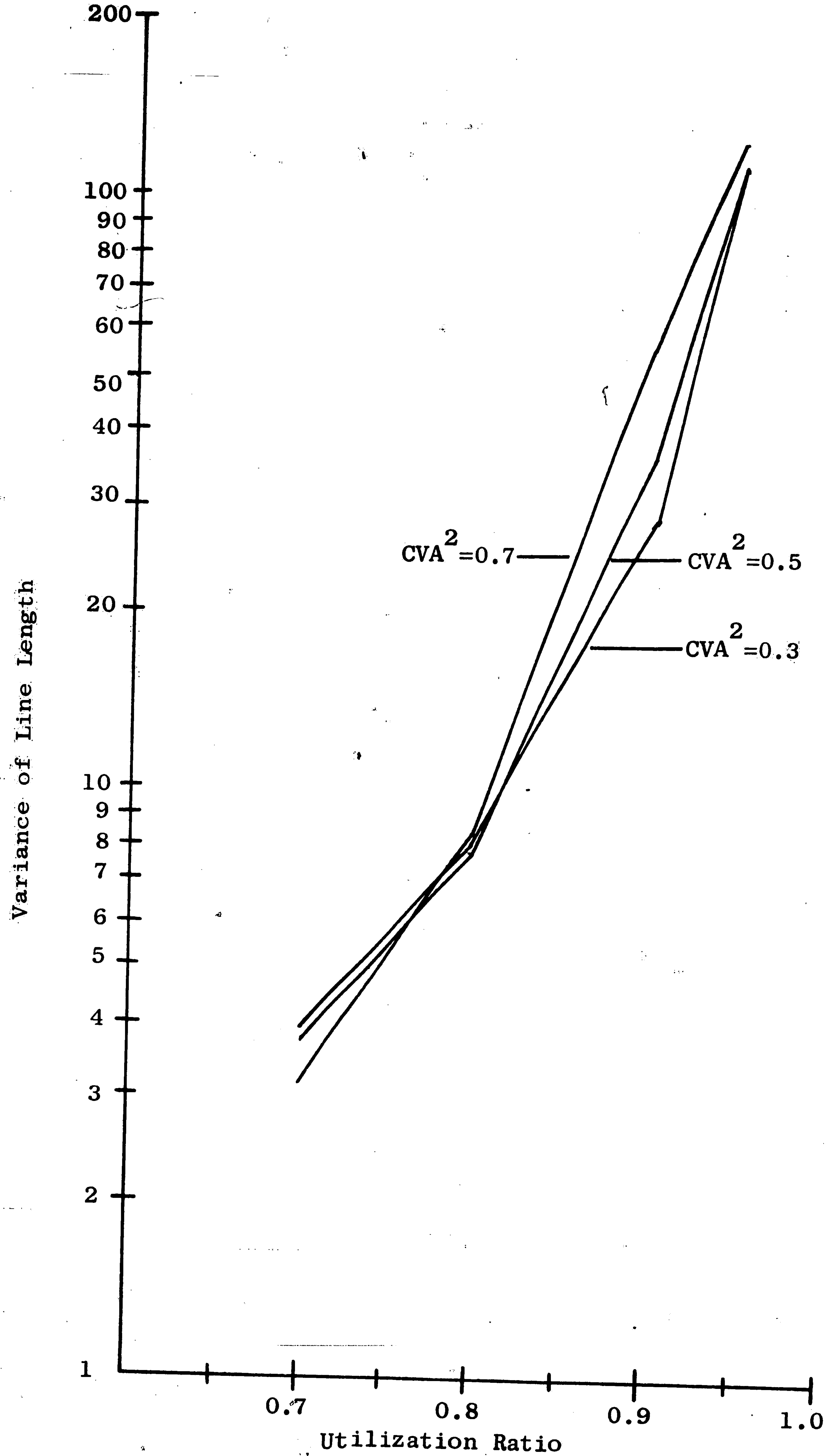

Figure 17 Variance of Line Length for N/M/1 System

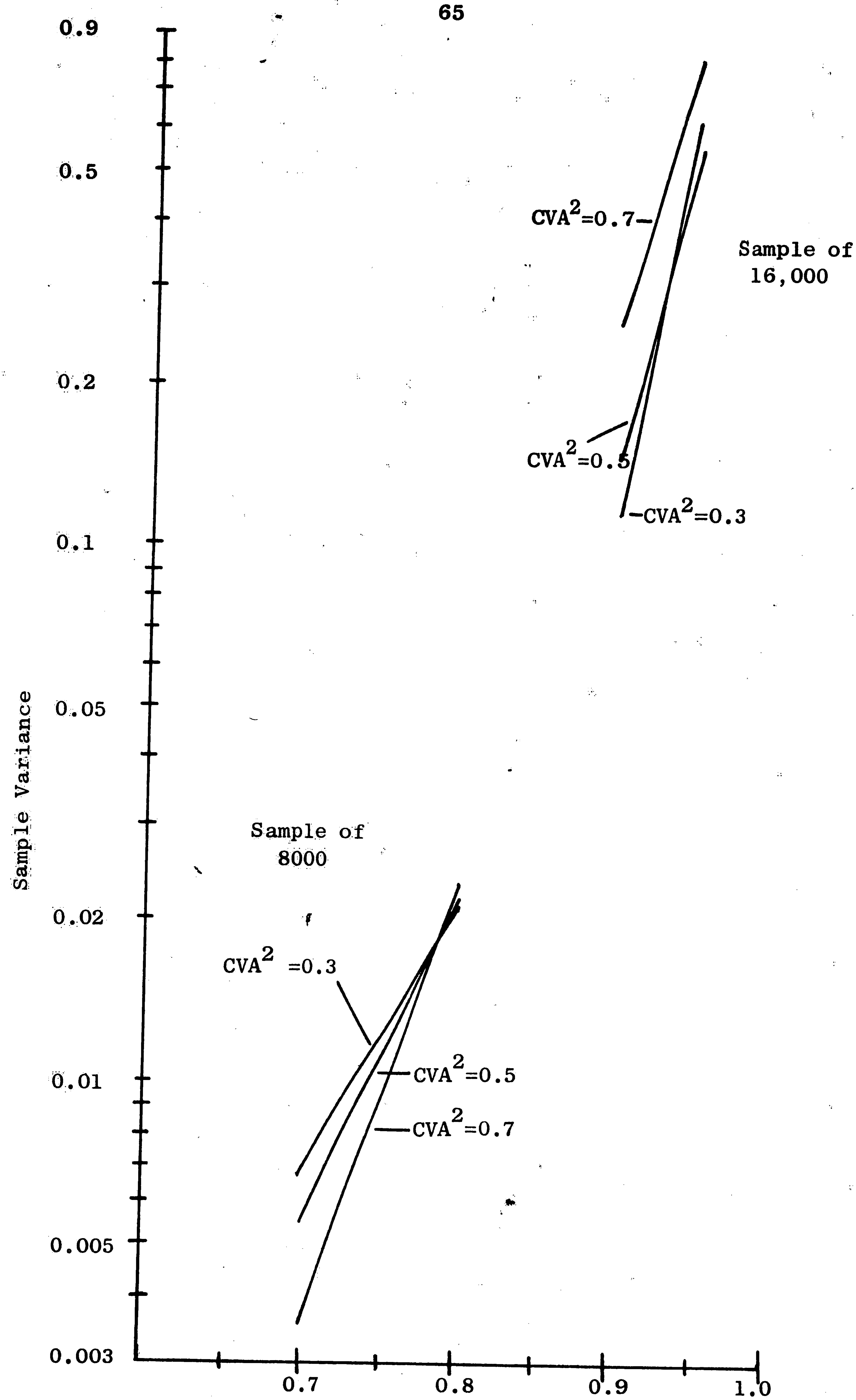

Figure 18 Sample Variance for N/M/1 System

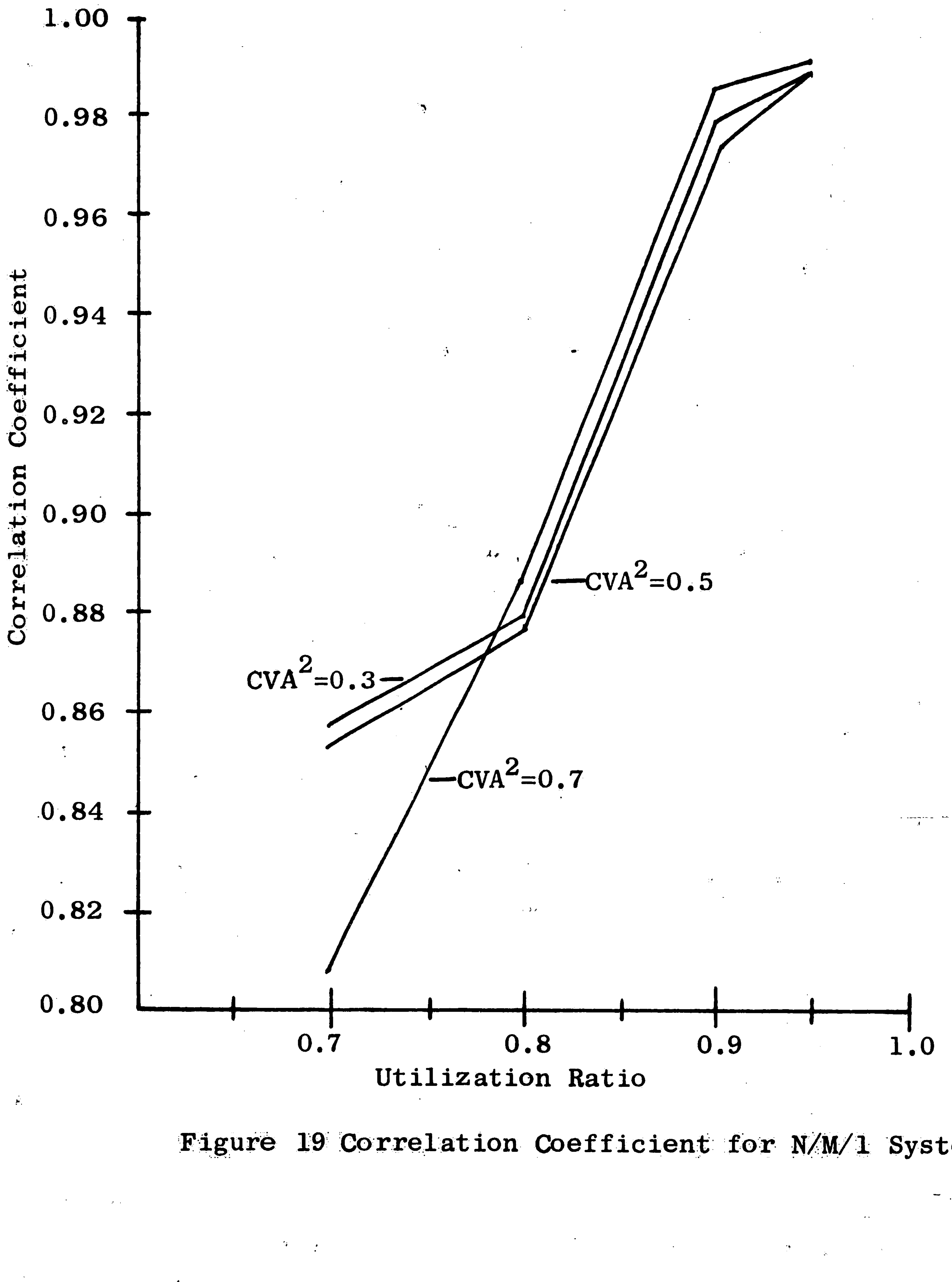

Figure 19 Correlation Coefficient for N/M/1 System

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		$CVA^2=0.3$		
0.7	1.72	3.80	0.003	0.769
0.8	2.88	9.57	0.020	0.901
0.9	6.32	42.55	0.139	0.976
0.95	13.50	187.96	0.991	0.994
		$CVA^2=0.5$		
0.7	1.92	4.93	0.006	0.816
0.8	3.23	12.45	0.033	0.992
0.9	7.15	55.37	0.205	0.982
0.95	15.21	240.70	1.316	0.995
		$CVA^2=0.7$		
0.7	2.07	5.89	0.008	0.843
0.8	3.50	14.96	0.047	0.935
0.9	7.81	66.72	0.268	0.985
0.95	16.56	286.82	1.60	0.996

Figure 20

Calculated Properties for Normal Arrivals

APPENDIX VI

RESULTS OF

M/W/1 SIMULATIONS CVS<1

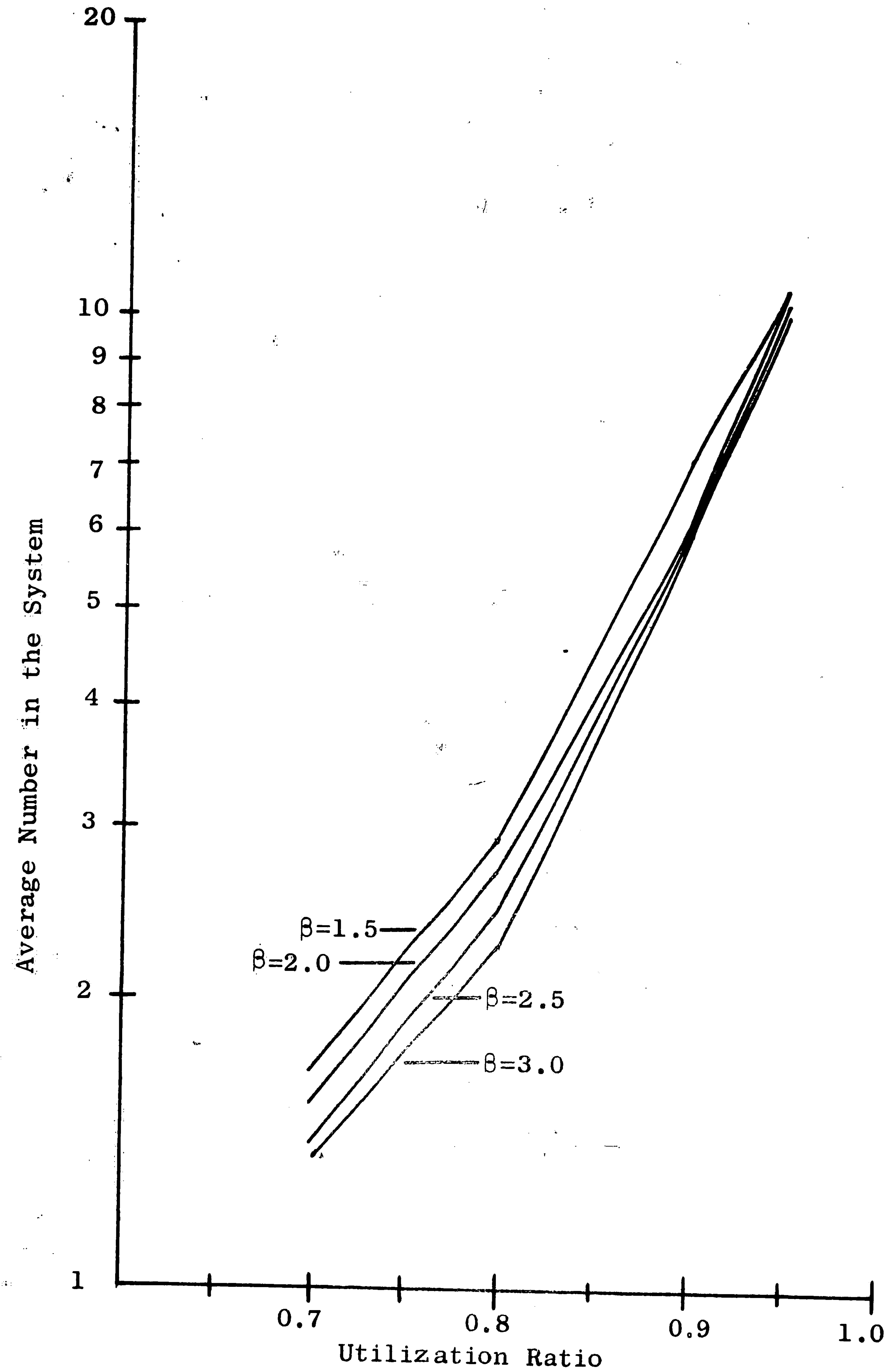

Figure 21 Line Length for M/W/1 System CVS<1

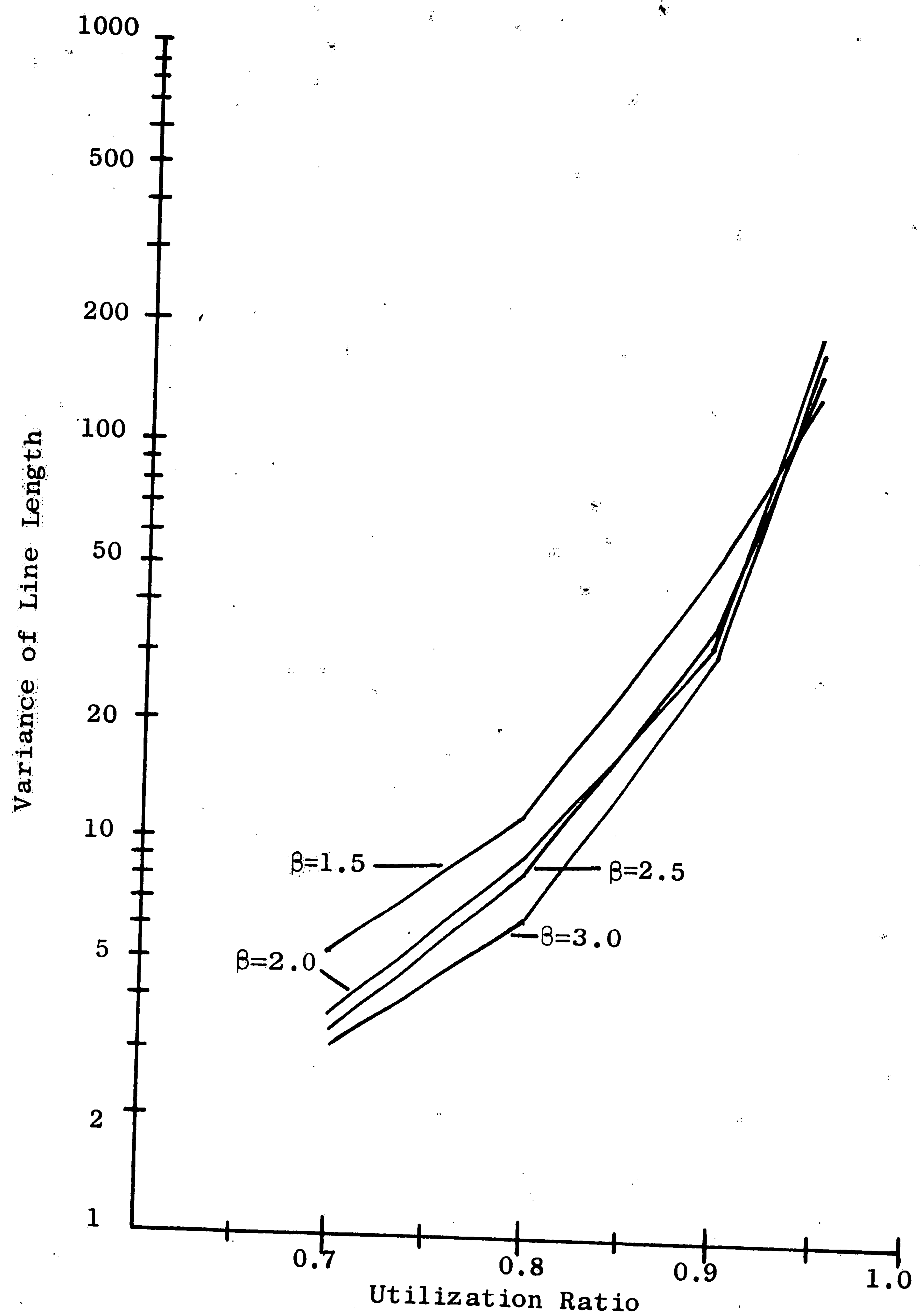

Figure 22 Variance of Line Length for M/W/1 System CVS<1

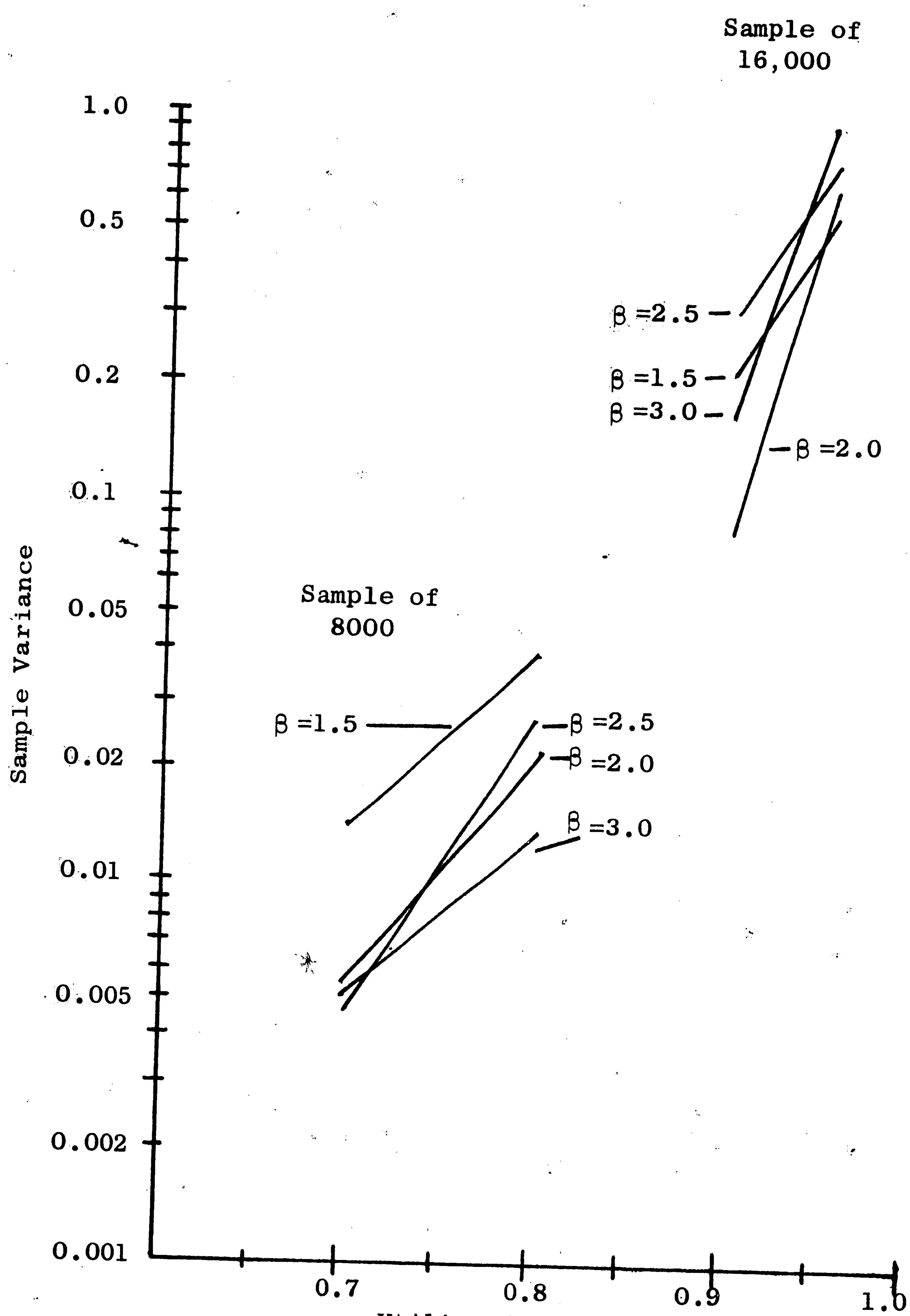

Figure 23 Sample Variance for M/W/1 System CVS<1

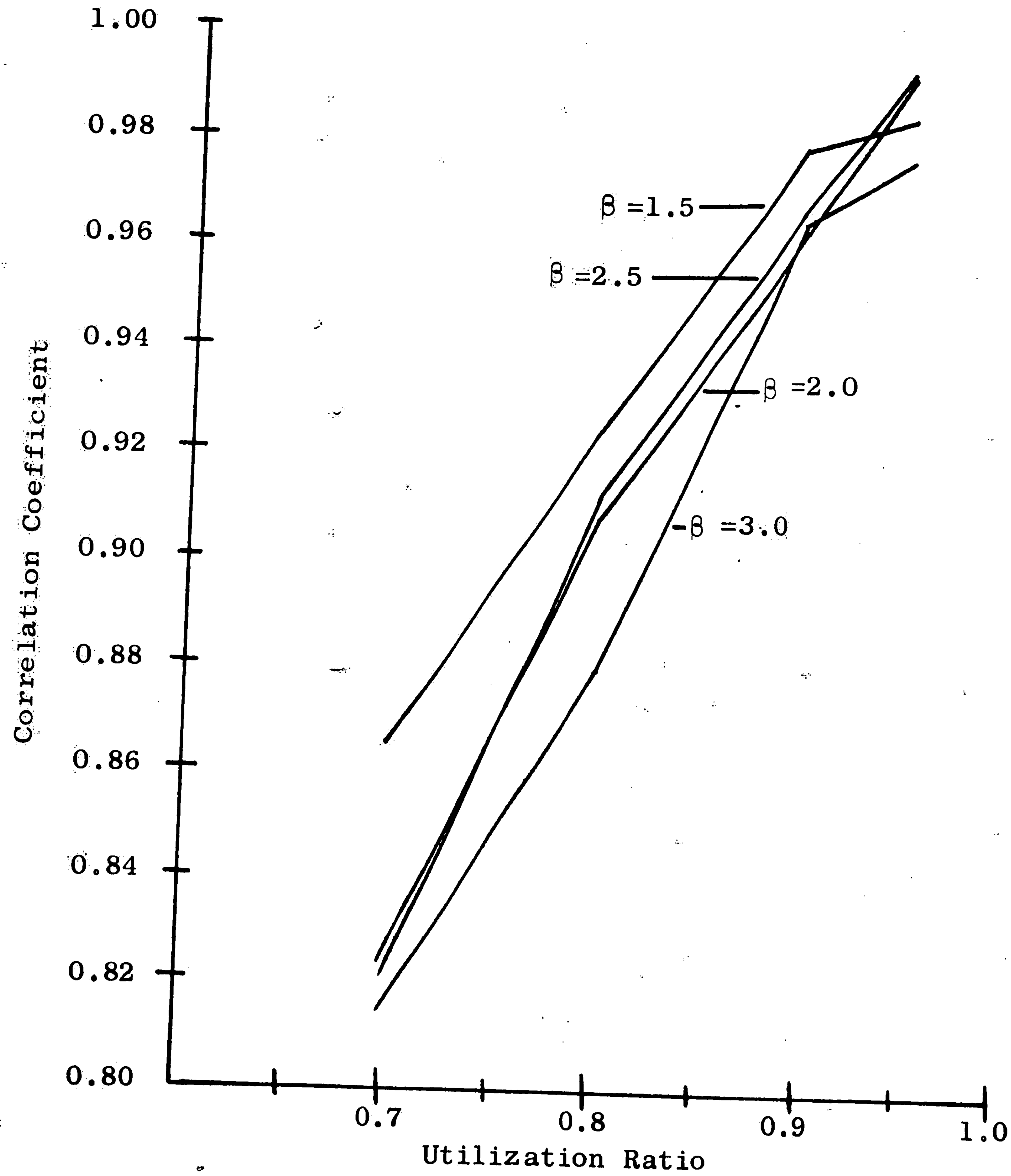

Figure 24 Correlation Coefficient for M/W/1 System CVS<1

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		$\beta = 1.5$		
0.7	1.89	3.58	0.003	0.756
0.8	3.13	9.22	0.019	0.897
0.9	6.81	41.48	0.133	0.976
0.95	14.13	175.17	0.913	0.994
		$\beta = 2.0$		
0.7	1.74	2.13	0.001	0.624
0.8	2.84	5.46	0.007	0.832
0.9	6.05	24.59	0.057	0.960
0.95	12.44	103.83	0.482	0.990
		$\beta = 2.5$		
0.7	1.67	1.42	0.0005	0.495
0.8	2.69	3.66	0.003	0.761
0.9	5.69	16.48	0.028	0.941
0.95	11.63	69.58	0.284	0.985
		$\beta = 3.0$		
0.7	1.62	1.02	0.0002	0.377
0.8	2.61	2.64	0.002	0.684
0.9	5.48	11.88	0.015	0.919
0.95	11.17	50.19	0.178	0.980

Figure 25

Calculated Properties for Weibull Service Times

CVS $<$ 1

APPENDIX VII

RESULTS OF

M/W/1 SIMULATIONS CVS>1

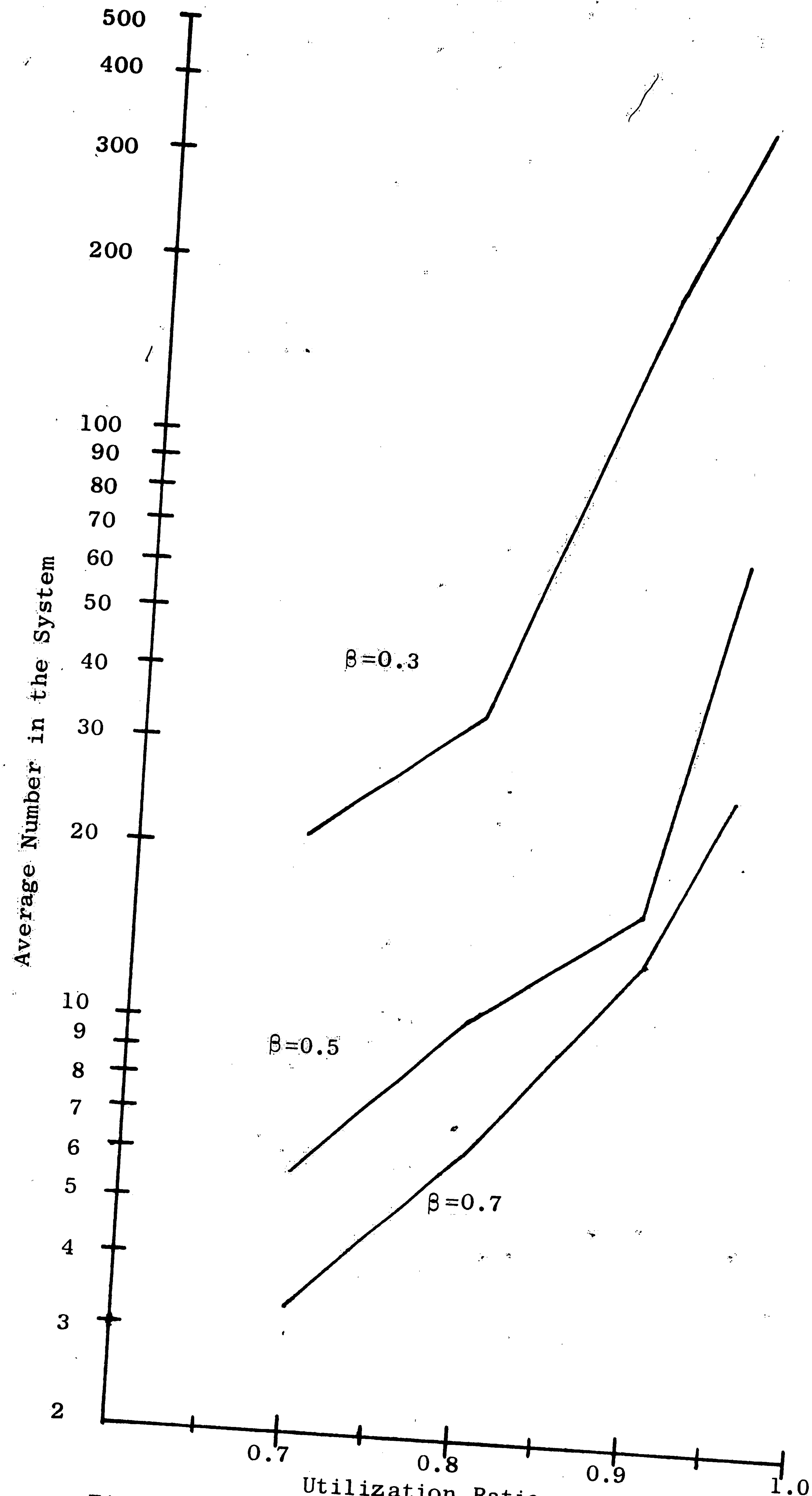

Figure 26 Line Length for M/W/1 System CVS > 1

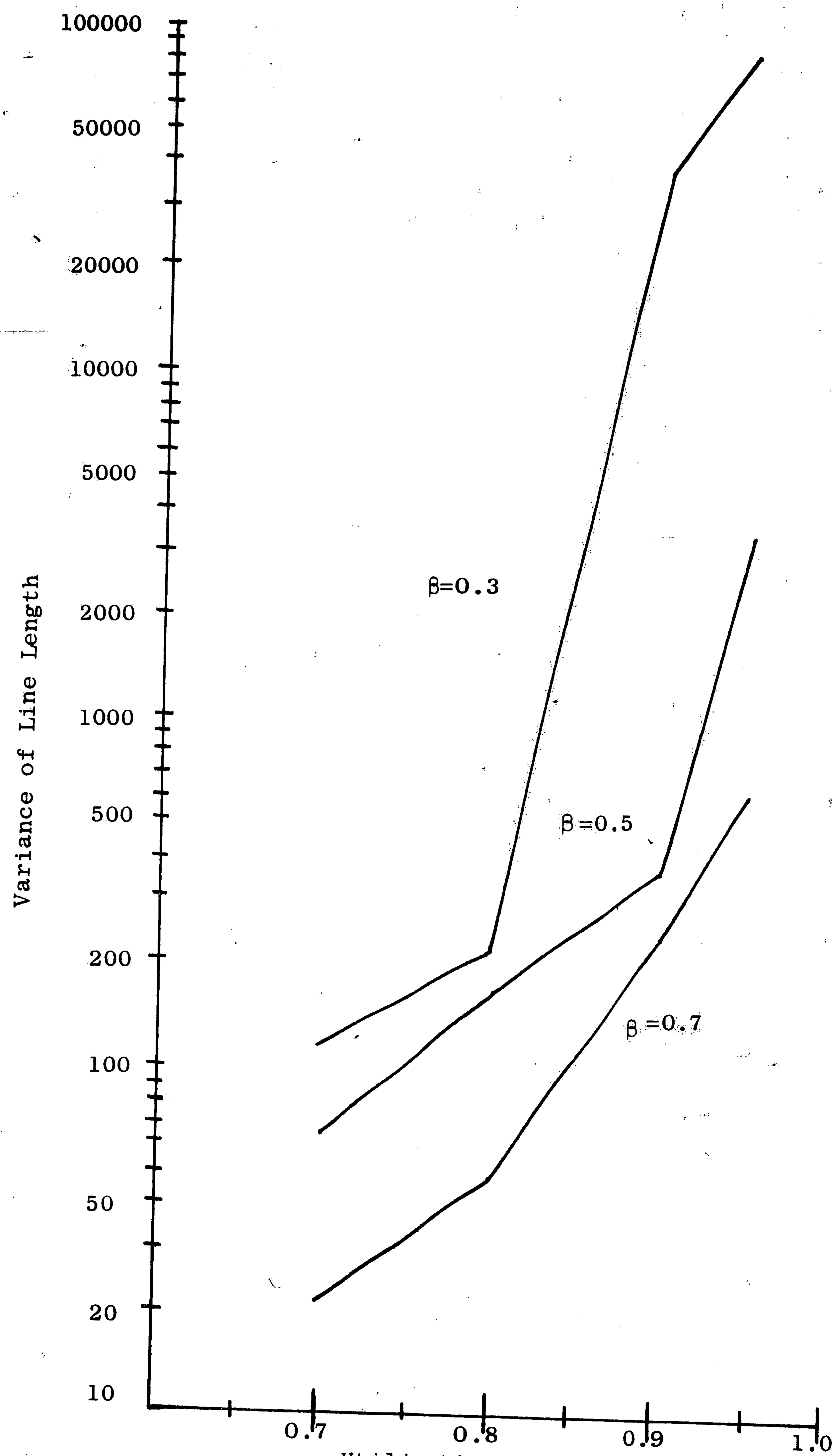

Figure 27 Variance of Line Length for M/W/1 System CVS > 1

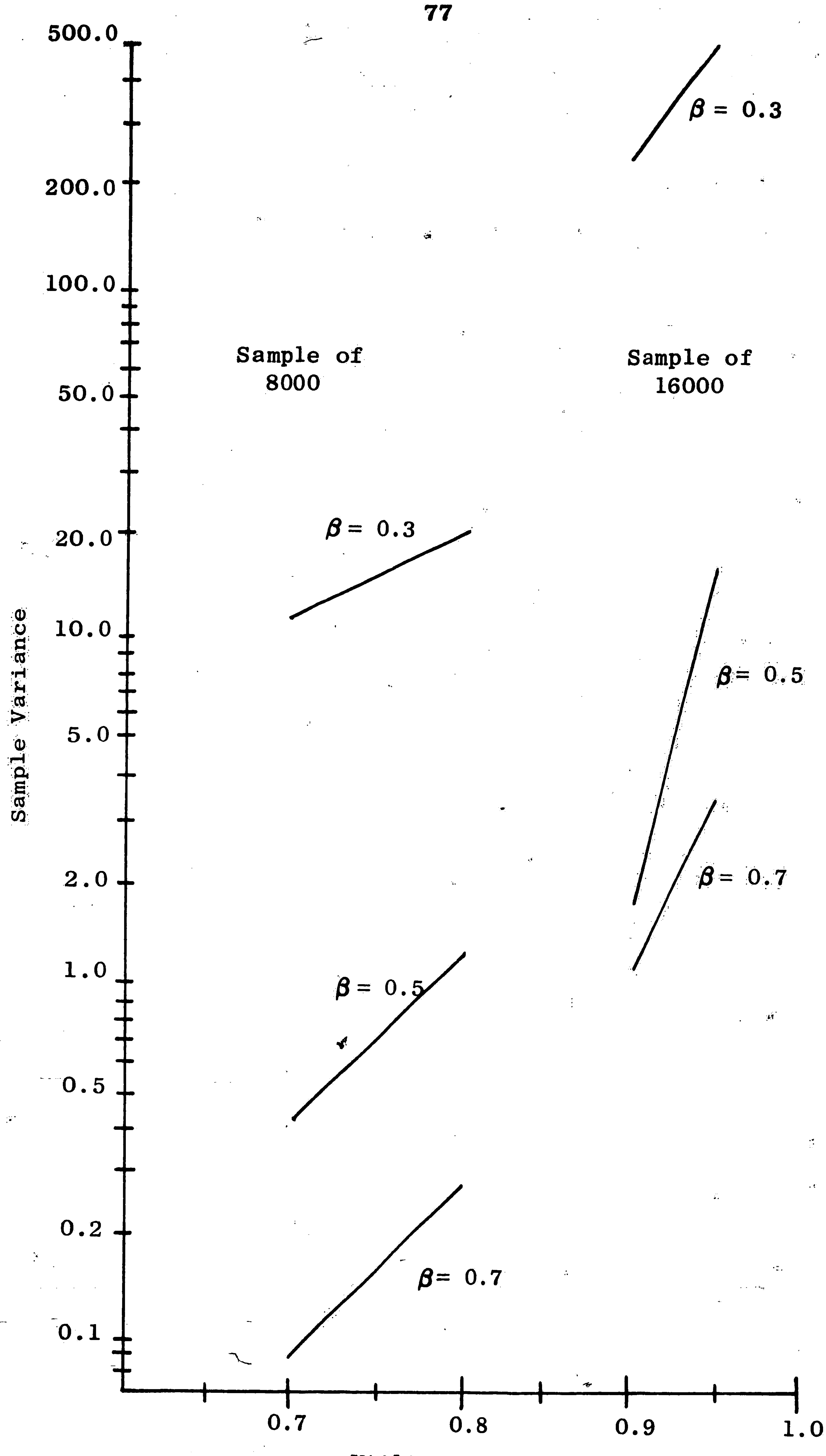

Figure 28 Sample Variance for M/W/1 System CVS $>$ 1

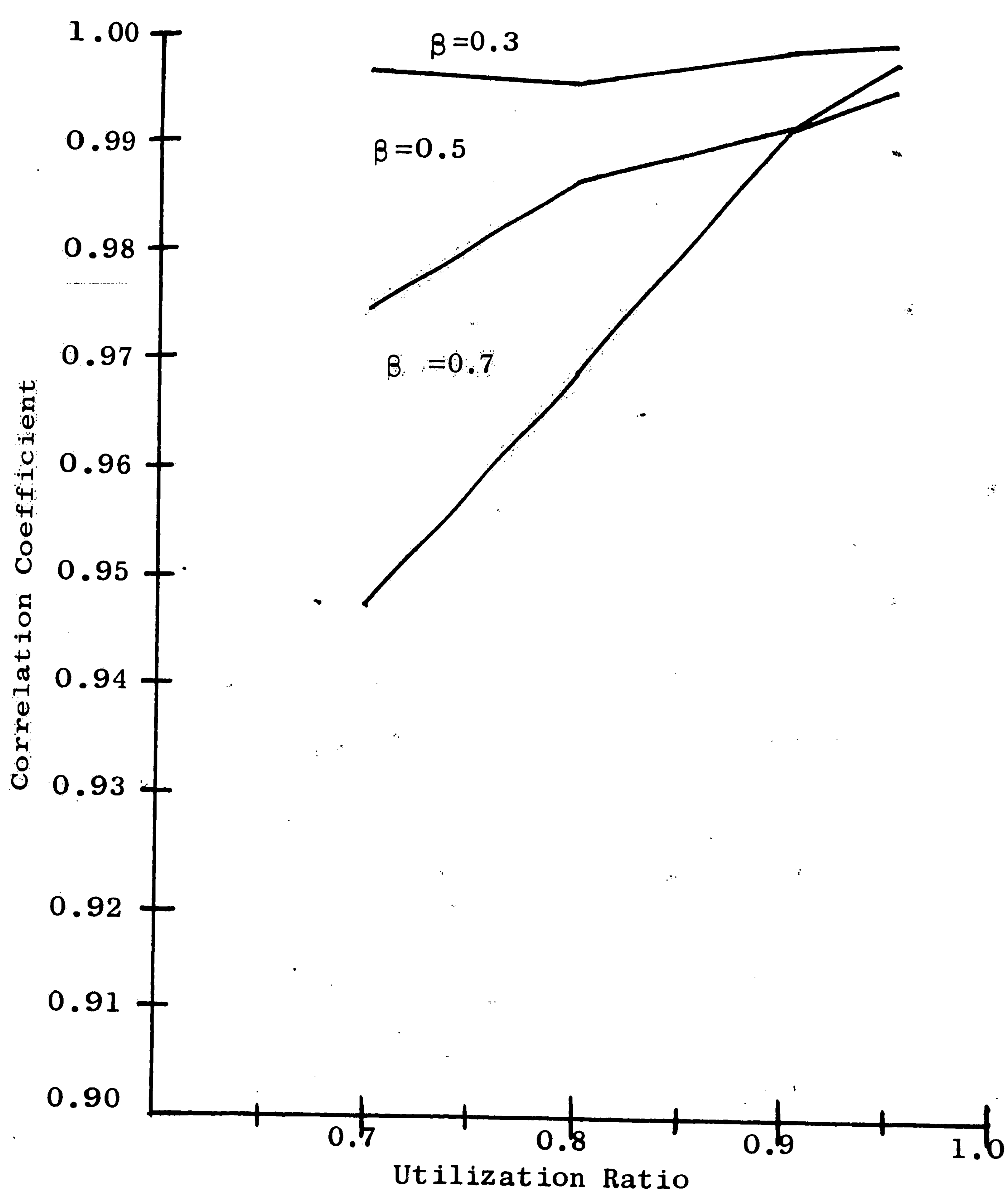

Figure 29 Correlation Coefficient for M/W/1 System CVS $>$1

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		$\beta=0.3$		
0.7	25.39	227.44	2.46	0.995
0.8	49.18	584.86	6.91	0.998
0.9	123.38	2631.87	16.24	0.9996
0.95	273.89	11112.32	69.24	0.9999
		$\beta=0.5$		
0.7	5.60	38.88	0.24	0.974
0.8	10.40	99.99	0.92	0.990
0.9	25.19	449.99	2.61	0.998
0.95	55.09	1899.99	11.66	0.9994
		$\beta=0.7$		
0.7	3.26	16.63	0.06	0.941
0.8	5.82	42.77	0.28	0.976
0.9	13.61	192.48	1.02	0.994
0.95	29.27	812.69	4.88	0.999

Figure 30

Calculated Properties for Weibull Service Times

CVS>1

APPENDIX VIII

RESULTS OF

M/N/1 SIMULATIONS

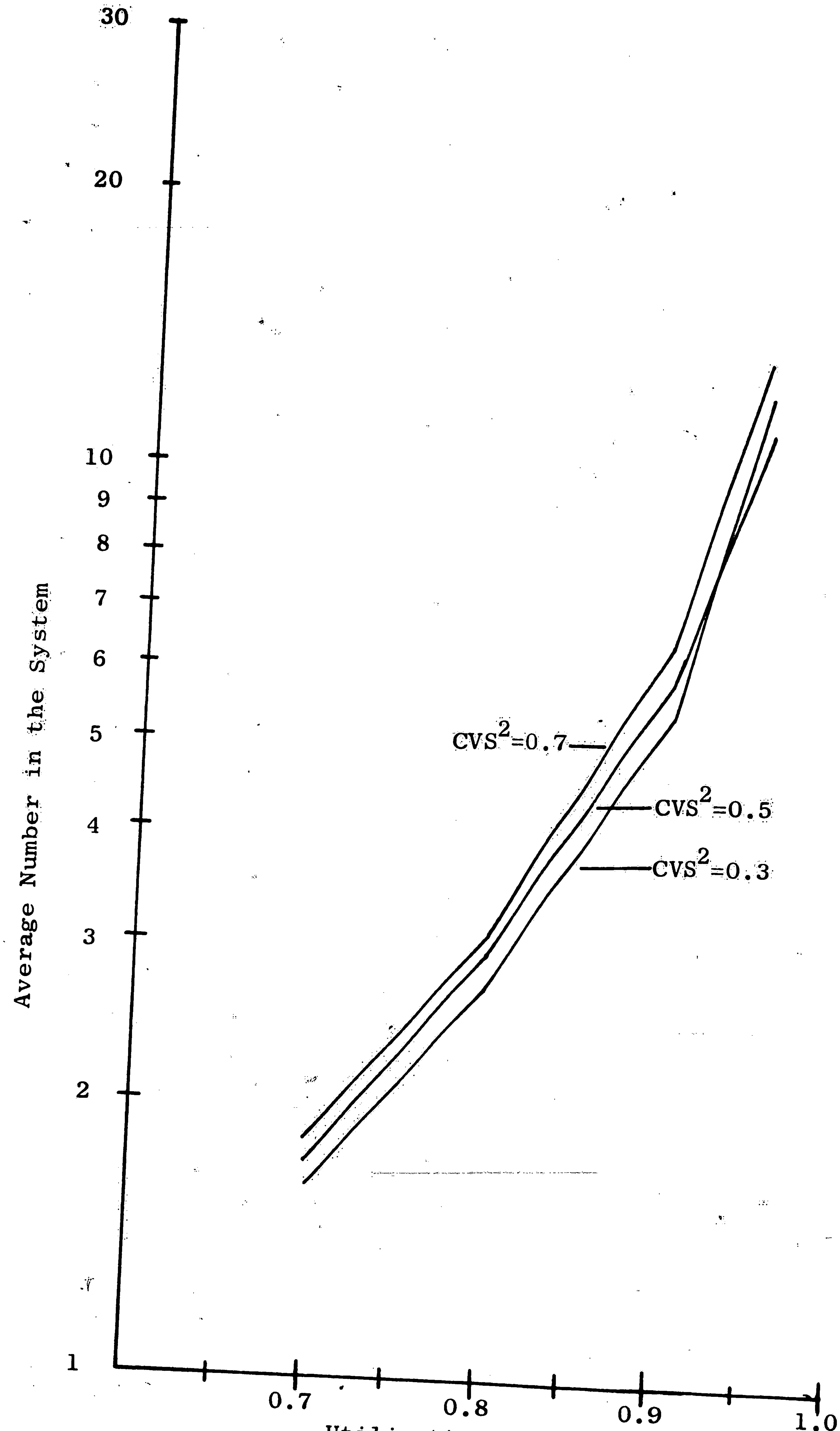

Figure 31 Line Length for M/N/1 System

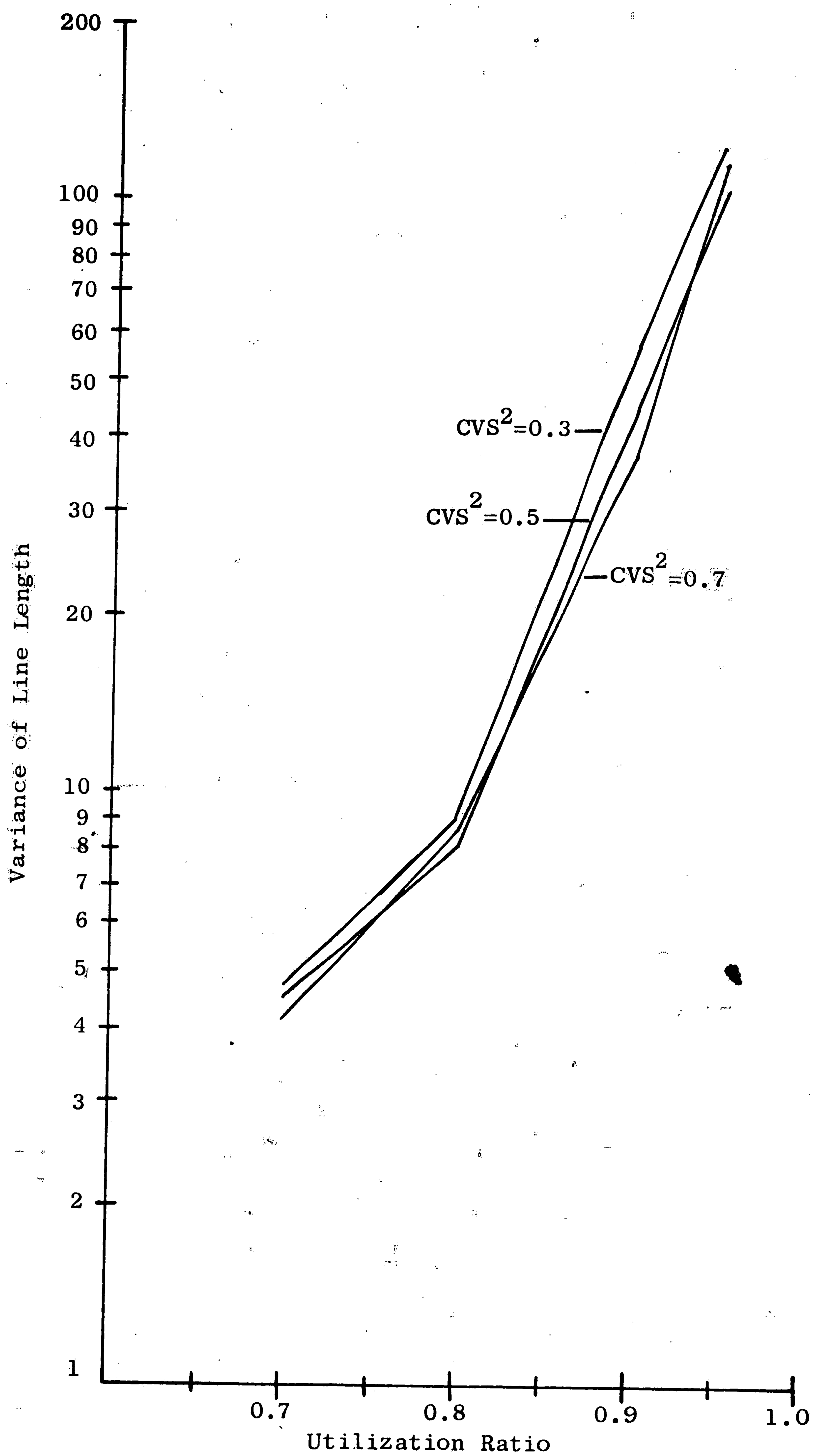

Figure 32 Variance of Line Length for M/N/1 System

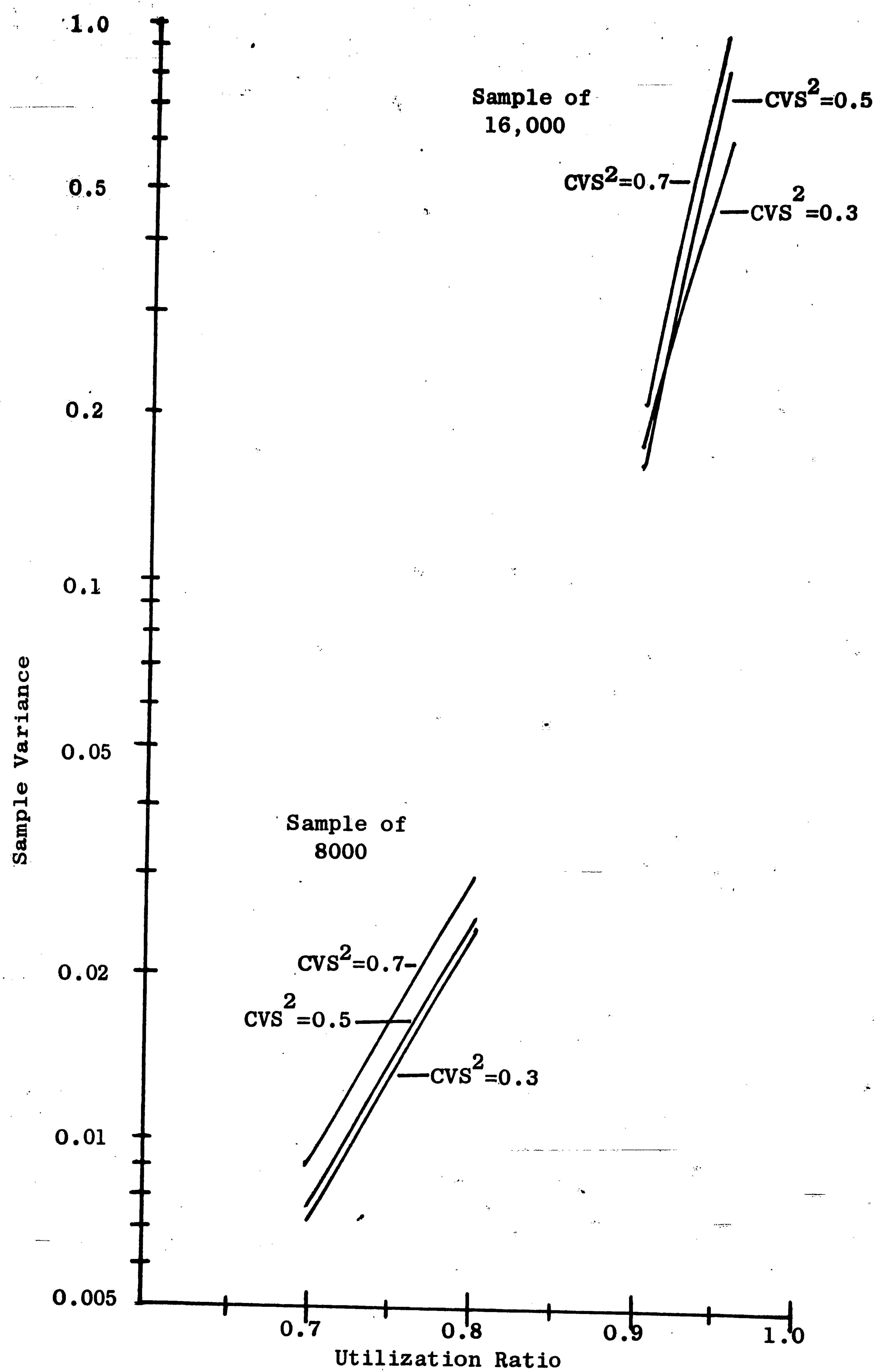

Figure 33 Sample Variance for M/N/l System

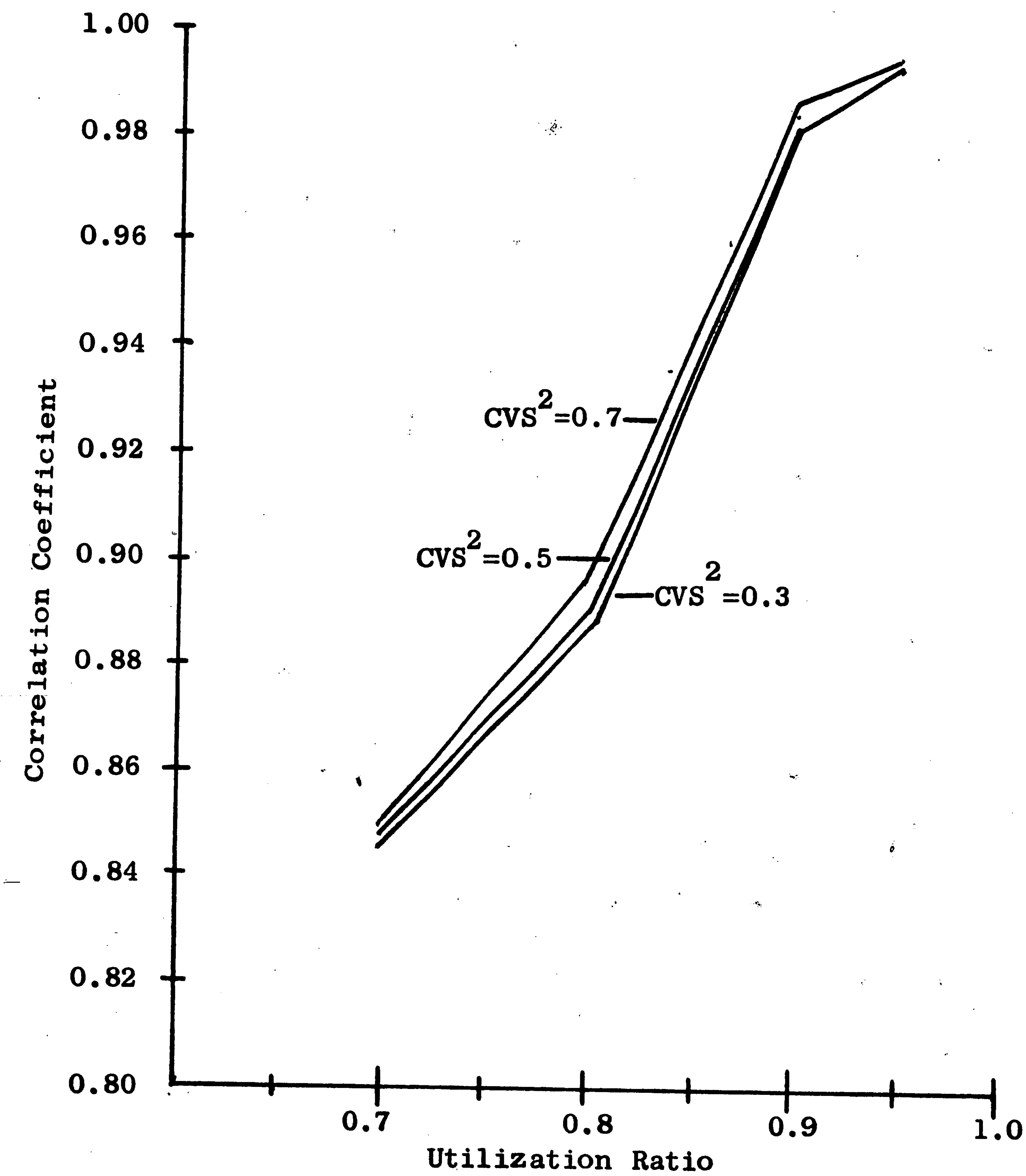

Figure 34 Correlation Coefficient for M/N/1 System

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		CVS^2=0.3		
0.7	1.76	2.33	0.001	0.651
0.8	2.88	5.99	0.008	0.846
0.9	6.16	26.97	0.067	0.963
0.95	12.68	113.89	0.542	0.991
		CVS^2=0.5		
0.7	1.94	4.07	0.004	0.782
0.8	3.23	10.47	0.024	0.908
0.9	7.07	47.14	0.162	0.979
0.95	14.70	199.05	1.059	0.995
		CVS^2=0.7		
0.7	2.08	5.45	0.007	0.832
0.8	3.52	14.01	0.042	0.931
0.9	7.78	63.03	0.248	0.996
0.95	16.29	266.15	1.473	

Figure 35

Calculated Properties for Normally Distributed Service Times

APPENDIX IX

RESULTS OF

W/W/1 SIMULATIONS

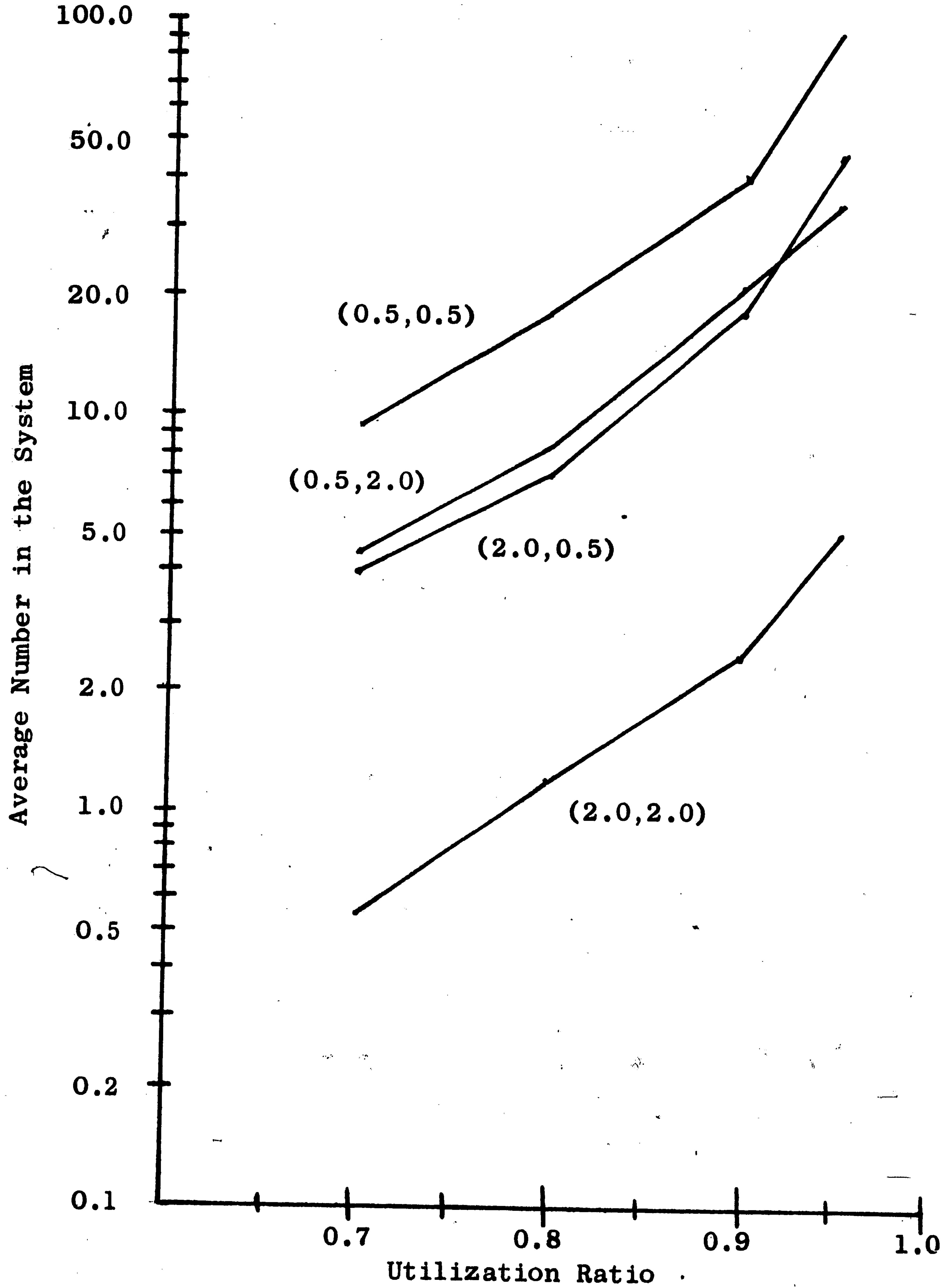

Figure 36 Line Length for W/W/1 System

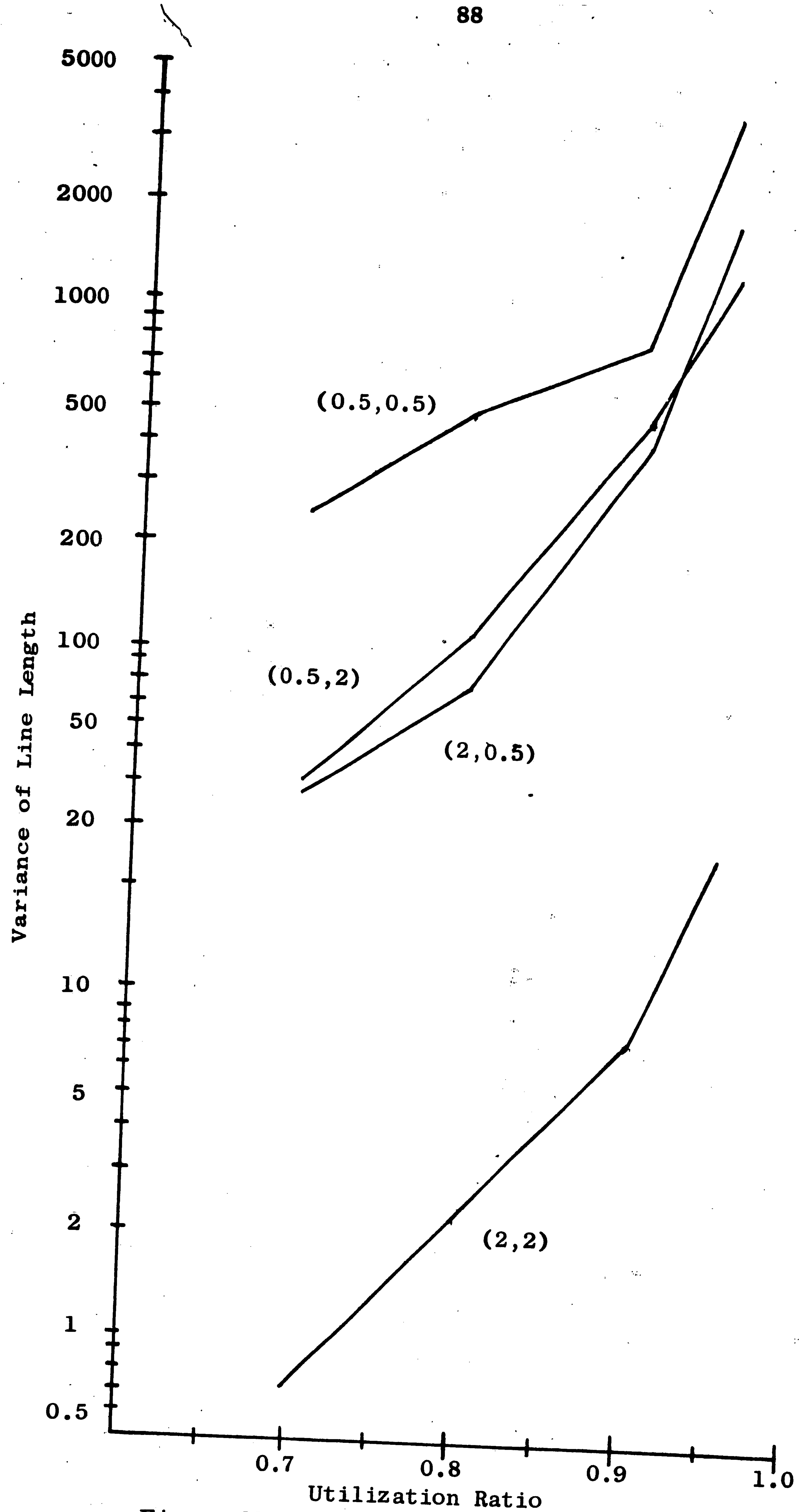

Figure 37 Variance of Line Length for W/W/1 System

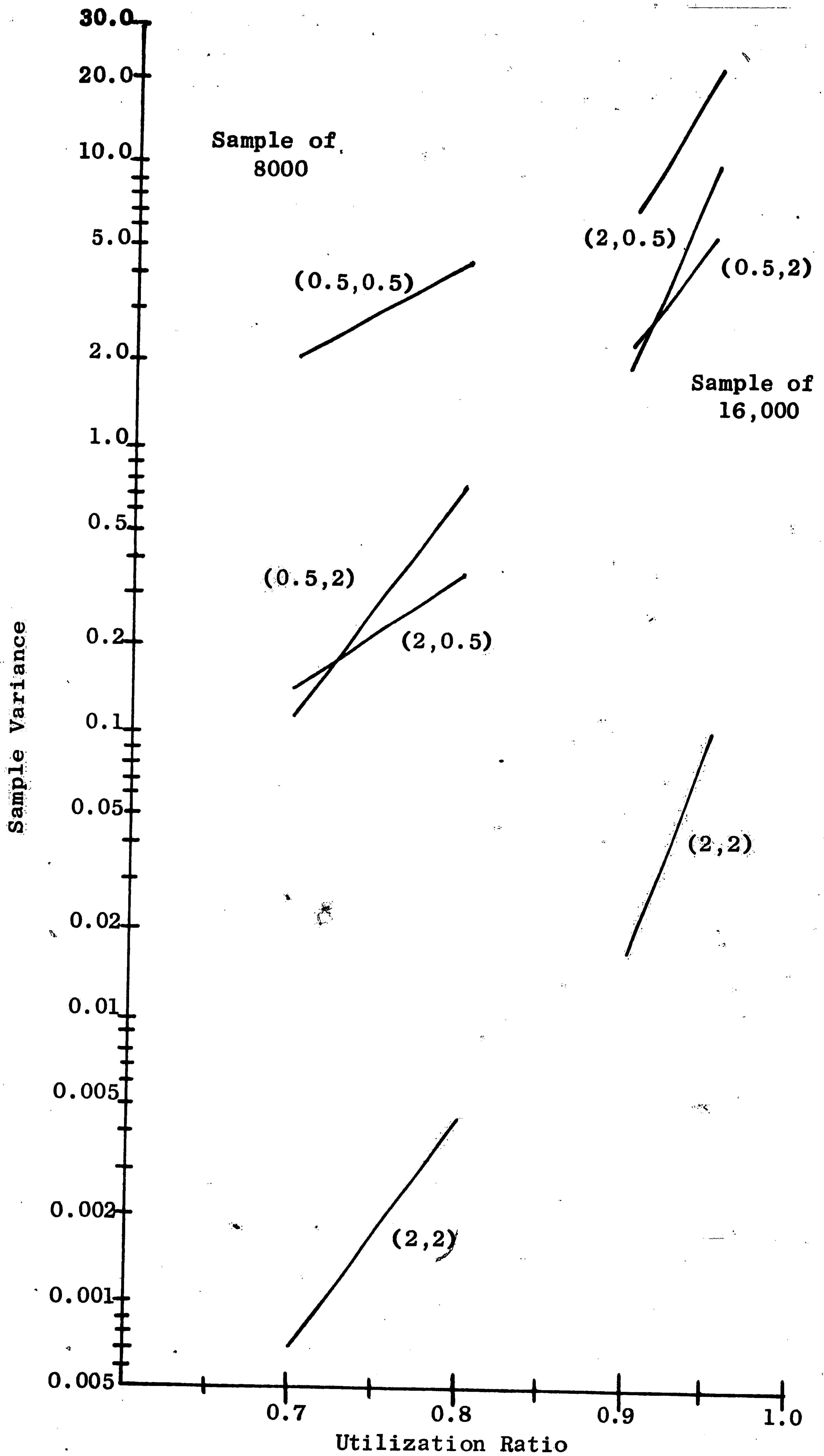

Figure 38 Sample Variance for W/W/1 System

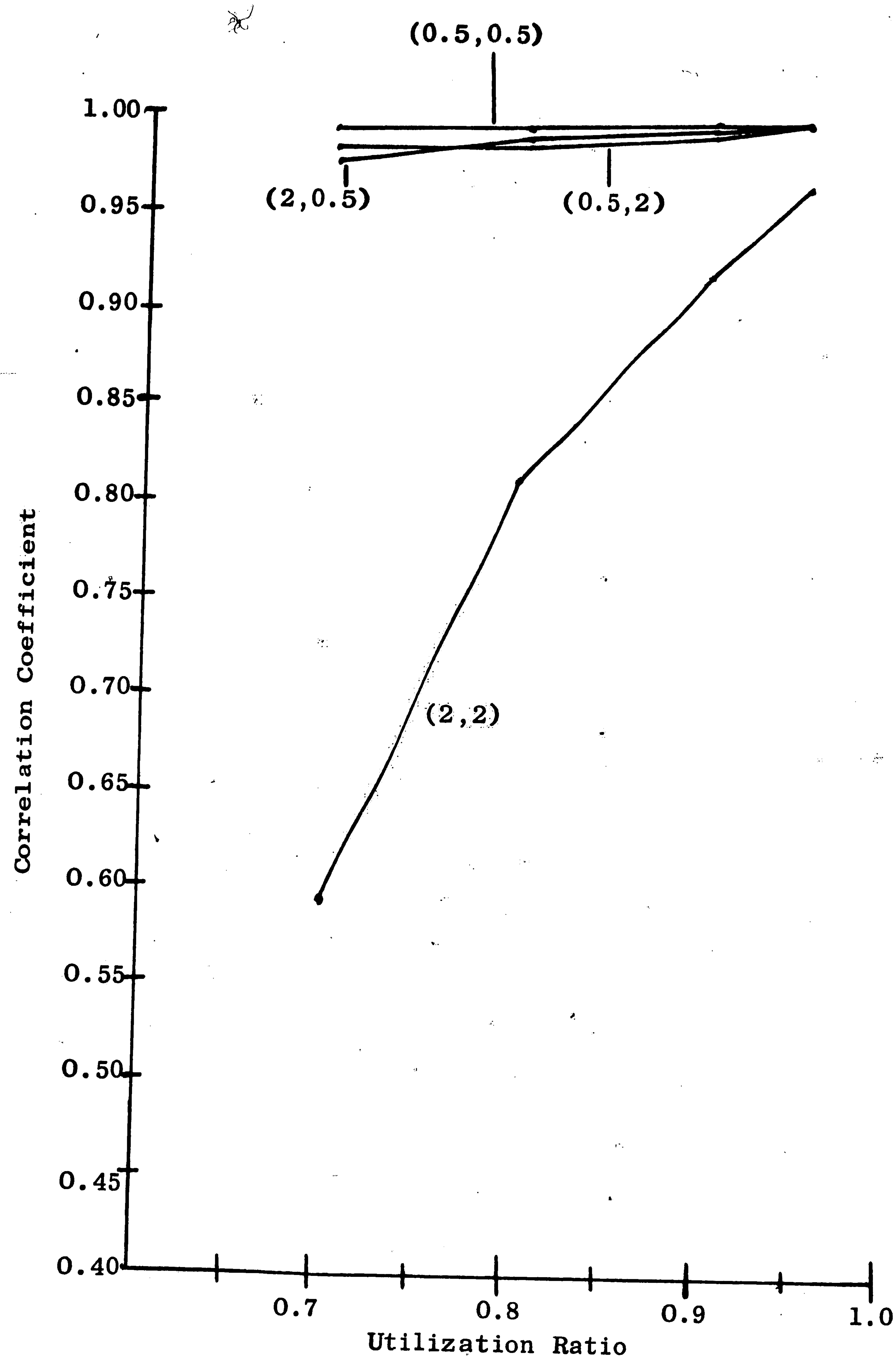

Figure 39 Correlation Coefficient for W/W/1 System

Utilization Factor	Average Line Length	Variance of Line Length	Sample Variance	Correlation Coefficient
		$(\beta_s,\beta_a)=(2.0,2.0)$		
0.7	1.53	2.85	0.001	0.704
0.8	2.48	6.78	0.010	0.863
0.9	5.24	28.33	0.073	0.965
0.95	10.67	115.13	0.549	0.991
		$(\beta_s,\beta_a)=(2.0,0.5)$ and $(0.5,2,0)$		
0.7	2.65	10.40	0.024	0.908
0.8	4.62	27.51	0.138	0.964
0.9	10.60	126.75	0.619	0.992
0.95	22.57	540.56	3.179	0.998
		$(\beta_s,\beta_a)=(0.5,0.5)$		
0.7	15.41	425.77	4.928	0.997
0.8	32.42	1544.51	18.896	0.9993
0.9	84.16	8572.30	53.371	0.9998
0.95	187.93	38849.30	242.588	0.9999

Figure 40

Calculations for the Case Where Both Service

and Arrivals are Non-exponential

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